Chapter4 Time Domain Analysis of Control System

- Ø Routh stability criterion
- ØSteady state errors
- ØTransient response of the first-order system
- Ø Transient response of the second-order system
- Ø Time domain performance specifications
- Ø The relationship between the performance specifications and system parameters
- Ø Transient response of higher-order systems

1

Definition of characteristic equation

$$\frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \mathbf{L} + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \mathbf{L} + a_1 s + a_0} = G(s)$$

The characteristic equation of the system is defined as

$$a_n s^n + a_{n-1} s^{n-1} + \mathbf{L} + a_1 s + a_0 = 0$$

2

Transfer function:

$$G(s)=\frac{C(s)}{R(s)}=\frac{k\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}[s+(s_k+jw_k)][(s+(s_k-jw_k)]}$$

z_i : zeros of the closed loop

$-p_j$, $-s_k \pm jw_k = -z_k w_n \pm jw_n\sqrt{1-z_k^2}$ poles of the closed loop

3

Step response:

$$C(s)=\frac{k\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}[s+(s_k+jw_k)][(s+(s_k-jw_k)]}\cdot\frac{1}{s}$$

$$C(s)=\frac{a_0}{s}+\sum_{j=1}^{q}\frac{a_j}{s+p_j}+\sum_{k=1}^{r}\frac{a_k s+b_k}{\left[s+(z_k w_n+jw_n\sqrt{1-z_k^2})\right]\left[s+(zw_n-jw_n\sqrt{1-z_k^2})\right]}$$

$$C(t)=a_0+\sum_{j=1}^{q}a_j e^{-p_j t}+\sum_{k=1}^{r}e^{-z_k w_k t}(B_k\cos w_k\sqrt{1-z_k^2}\,t+C_k\sin w_k\sqrt{1-z_k^2}\,t)$$
$$=a_0+\sum_{j=1}^{q}a_j e^{-p_j t}+\sum_{i=1}^{r}D_k e^{-z_k w_k t}\sin(w_k\sqrt{1-z_k^2}\,t+j_k)$$

4

4.1 Routh stability criterion

Consider that the characteristic equation of a LTI system

$$F(s) = a_n s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 = 0$$

Where all the coefficients are real numbers.

In order that there be no roots of the above equation with positive real parts, it is necessary but not sufficient that

1. All the coefficients of the polynomial have the same sign.
2. None of the coefficients vanishes.

5

The Routh tabulation

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1}s + a_n = 0$$

for n = 6

s^6	a_0	a_2	a_4	a_6
s^5	a_1	a_3	a_5	0
s^4	$\frac{a_1a_2 - a_0a_3}{a_1} = A$	$\frac{a_1a_4 - a_0a_5}{a_1} = B$	$\frac{a_1a_6 - a_0 \times 0}{a_1} = a_6$	0
s^3	$\frac{Aa_3 - a_1B}{A} = C$	$\frac{Aa_5 - a_1a_6}{A} = D$	$\frac{A\times 0 - a_1\times 0}{A} = 0$	0
s^2	$\frac{BC - AD}{C} = E$	$\frac{Ca_6 - A\times 0}{C} = a_6$	$\frac{C\times 0 - A\times 0}{C} = 0$	0
s^1	$\frac{ED - Ca_6}{E} = F$	0	0	0
s^0	$\frac{Fa_6 - E\times 0}{F}$	0	0	0

6

Rout-Hurwitz criterion

The roots of the polynomial are all in the left half of the s-plane if all the elements of the first column of the Routh Array are of the same sign.

If there are changes of signs in the elements of the first column, the number of sign changes indicates the number of roots with positive real parts.

The necessary and sufficient condition for the stability of a system

7

Example: The characteristic equation of a system is

$$s^3 + 4s^2 + 10s + 50 = 0$$

Determine the stability of the system using Routh criterion.

Solution: （1）. Check the necessary condition

（2). Routh array is

s^3	1	10
s^2	4	50
s^1	-2.5	0
s^0	50	0

The system has two roots located in the right half of the s-plane.

8

Special case 1

The first element in any one row of the Routh Array is zero but the other elements are not.

We can replace the zero element in the Routh tabulation by an arbitrary small positive number ε and then proceed with the Routh test.

9

Example: The characteristic equation of a system is

$$s^5 + s^4 + 5s^3 + 5s^2 + 2s + 1 = 0$$

Determine the stability of the system using Routh criterion.

Solution: Routh array is

s^5	1	5	2
s^4	1	5	1
s^3	$0(e)$	1	0
s^2	$\frac{5e-1}{e}$	1	0
s^1	$\frac{5e-1-e^2}{5e-1}$	0	0
s^0	1	0	0

There are two sign changes in the first column of the tabulation, the system has two roots located in the right half of the s-plane.

10

Special case 2

The elements in one row of the Routh Array are all zero.

11

Example: The characteristic equation of the system is:

$$s^3 + s^2 + 16s + 16 = 0$$

The Routh array is :

s^3	1	16
s^2	1	16
s^1	0	0

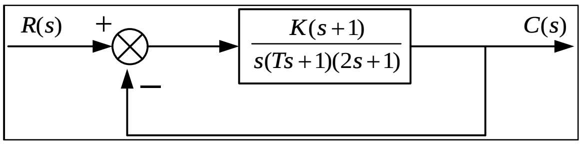
The auxiliary equation is: $A(s) = s^2 + 16 = 0$

s^3	1	16
s^2	1	16
s^1	2	0
s^0	16	0

The sign of the elements in the first column does not change, the system is stable .

12

Example:



Determine the value of K and T to make the closed loop be stable.

Solution: the characteristic equation:

$$2Ts^3 + (2+T)s^2 + (K+1)s + K = 0$$

Routh array:

s^3	$2T$	$K+1$
s^2	$2+T$	K
s^1	$\frac{(2+T)(K+1)-2TK}{2+T}$	0
s^0	K	0

13

$K > 0, 2T > 0,$ $T > 0$

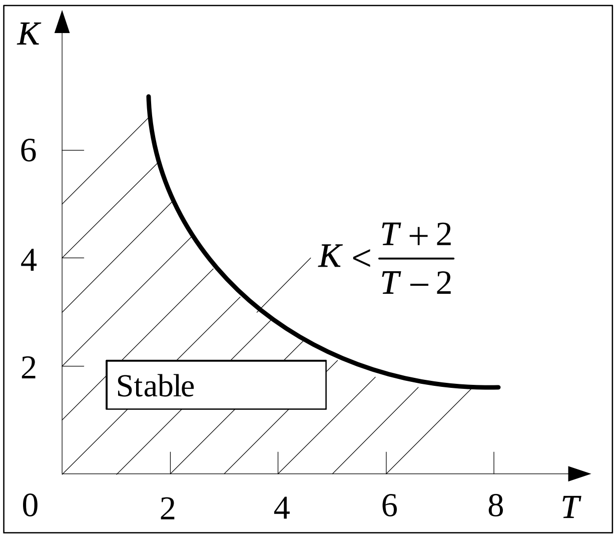
$2 + T > 0$

$(2+T)(K+1) - 2TK > 0$

The condition for the stability is:

$$0 < K < \frac{T+2}{T-2}$$

14



$$0 < K < \frac{T+2}{T-2}$$

15

Example: The characteristic equation of the system is

$$s^3 + 8s^2 + 10s + 2 = 0$$

Determine the stability of the system. Analyze how many roots lie between the imaginary axis and the line $s = -1$.

Solution: Routh array:

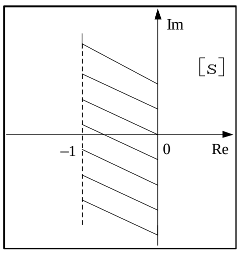
s^3	1	10
s^2	8	2
s^1	9.75	0
s^0	2	0

the system is stable.

Let $s = s_1 - 1$

The characteristic equation becomes:

$$s_1^3 + 5s_1^2 - 3s_1 - 1 = 0$$



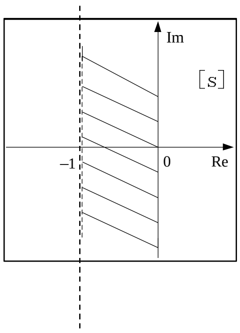
16

$$s_1^3 + 5s_1^2 - 3s_1 - 1 = 0$$

Routh array for the above equation:

s_1^3	1	−3
s_1^2	5	−1
s_1^1	−2.8	0
s_1^0	−1	0

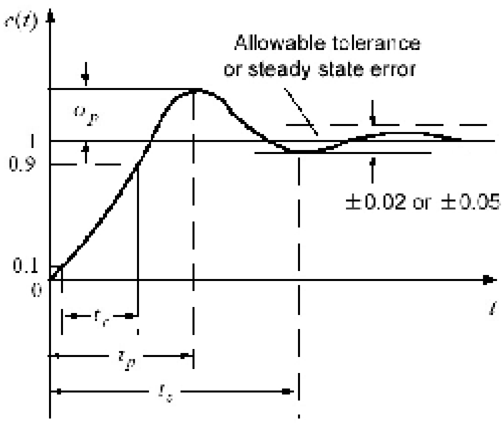
there is one root on the right side of the line s = -1.



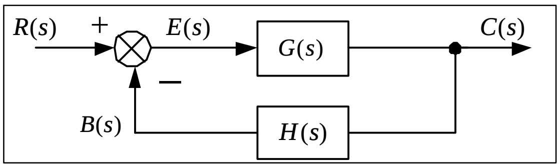
17

4.6 Steady state errors

$$e(\infty) = c(\infty) - r(\infty)$$



18



$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{1}{1+G(s)H(s)} R(s)$$

19

the types of the control systems

$$G(s) = \frac{K_r(s+z_1)(s+z_2)\cdots(s+z_m)}{s^n(s+p_1)(s+p_2)\cdots(s+p_q)} = \frac{K_r\prod_{i=1}^{m}(s+z_i)}{s^n\prod_{j=v+1}^{q}(s+p_j)}$$

ν = 0, type 0 system
ν = 1, type 1 system
ν = 2, type 2 system

20

a. position error constant

For a step input $r(t) = R \cdot u(t)$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{R(s)}{1+G(s)H(s)}$$
$$= \lim_{s\to 0} s\frac{R/s}{1+G(s)H(s)} = \lim_{s\to 0}\frac{R}{1+G(s)H(s)}$$

$$e_{ss} = \lim_{s\to 0}\frac{R}{1+G(s)H(s)} = \frac{R}{1+\lim_{s\to 0} G(s)H(s)}$$

$$\lim_{s\to 0} G(s)H(s) = K_P$$

21

b. velocity error constant

For a ramp input: $r(t) = Rt$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{R(s)}{1+G(s)H(s)}$$
$$= \lim_{s\to 0} s\frac{R/s^2}{1+G(s)H(s)} = \lim_{s\to 0}\frac{R}{sG(s)H(s)}$$

$$K_v = \lim_{s\to 0} sG(s)H(s)$$ $$e_{ss} = \frac{R}{K_v}$$

22

c. Acceleration error constant

for acceleration input $r(t)=\frac{1}{2}Rt^2$

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=\lim_{s\to 0}s\frac{R(s)}{1+G(s)H(s)}$$

$$=\lim_{s\to 0}s\frac{R/s^3}{1+G(s)H(s)}=\lim_{s\to 0}\frac{R}{s^2G(s)H(s)}$$

$$K_a=\lim_{s\to 0}s^2G(s)H(s)$$

$$e_{ss}=\frac{R}{K_a}$$

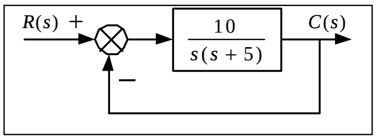
23

Steady state error

system type	input			
	impulse	step	ramp	acceleration
0	0	$\frac{R}{1+K_p}$	∞	∞
1	0	0	$\frac{R}{K_v}$	∞
2	0	0	0	$\frac{R}{K_a}$

24

Example: Determine the steady state errors, when the input is $u(t)$, t and $\frac{1}{2}t^2$.



Solution:

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{10}{s(s+5)} = \infty$$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} s\frac{10}{s(s+5)} = 2$$

$$K_a = \lim_{s\to 0} s^2G(s)H(s) = \lim_{s\to 0} s^2\frac{10}{s(s+5)} = 0$$

25

$$e_{ssp} = \frac{1}{1+K_P} = 0$$

$$e_{ssv} = \frac{1}{K_v} = \frac{1}{2}$$

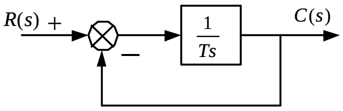
$$e_{ssa} = \frac{1}{K_a} = \infty$$

26

4.1 The transient response of the first-order system

4.1.1 The math model

$$\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$$



27

4.1.2 the step response

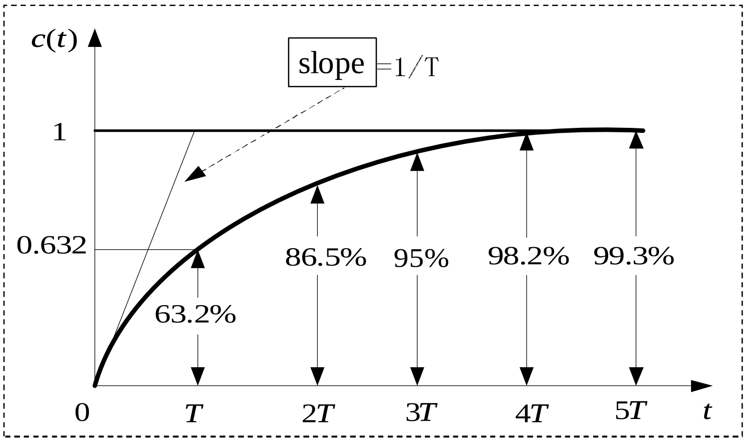
r(t) = u(t) , R(s)=1/s

$$C(s) = G(s)R(s) = \frac{1}{(Ts+1)} \cdot \frac{1}{s}$$

$$= \frac{1}{s} - \frac{T}{1+Ts}$$

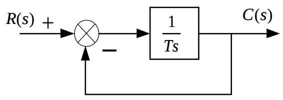
$$c(t) = 1 - e^{-\frac{t}{T}} \qquad t \geq 0$$

28



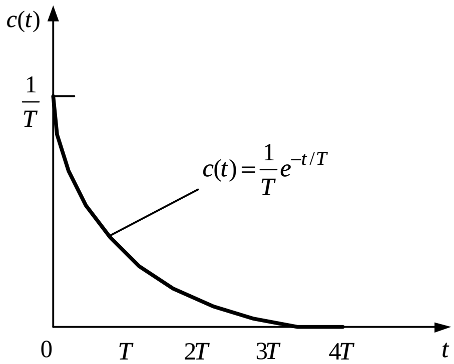
29

Impulse response:



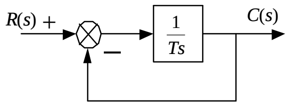
$$C(s) = \frac{1}{1+Ts} \bullet R(s) = \frac{1}{1+Ts}$$

$$g(t) = c(t) = \frac{1}{T} e^{-\frac{t}{T}}$$



30

Ramp response：

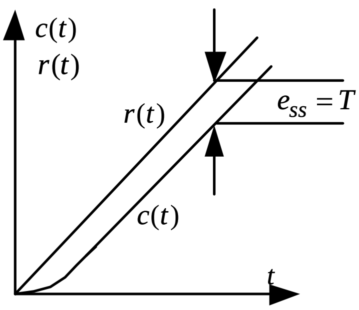


$$C(s) = G(s)R(s) = \frac{1}{1+Ts} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+1/T}$$

$$c(t) = t - T + Te^{-t/T}$$

31

$$c(t) = t - T + Te^{-t/T}$$

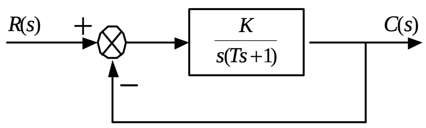


$$\lim_{t\to\infty}[c(t) - r(t)] = \lim_{t\to\infty}[t - T + Te^{-t/T} - t] = -T$$

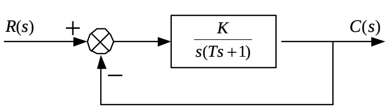
32

4.2 the transient response of the second-order system

4.2.1 the mathematical model



33



$$F(s) = \frac{C(s)}{R(s)} = \frac{\frac{K}{T}}{s^2 + \frac{1}{T}s + \frac{K}{T}}$$

let

$$w_n^2 = \frac{K}{T} \qquad 2Vw_n = \frac{1}{T}$$

$$F(s) = \frac{w_n^2}{s^2 + 2zw_n s + w_n^2}$$

34

The characteristic equation :

$$s^2 + 2zw_n s + w_n^2 = 0$$

The roots of the characteristic equation are

$$s_{1,2} = -zw_n \pm w_n\sqrt{z^2 - 1}$$

35

1. underdamped $(0 < z < 1)$

$$s_{1,2} = -zw_n \pm w_n\sqrt{z^2 - 1} \qquad s_{1,2} = -s \pm jw_d$$

If input is a unit step u(t):

$$C(s) = \frac{w_n^2}{(s^2 + 2zw_n s + w_n^2)} \cdot \frac{1}{s} = \frac{1}{s} - \frac{s + zw_n}{(s + zw_n)^2 + w_d^2} - \frac{zw_n}{(s + zw_n)^2 + w_d^2}$$

36

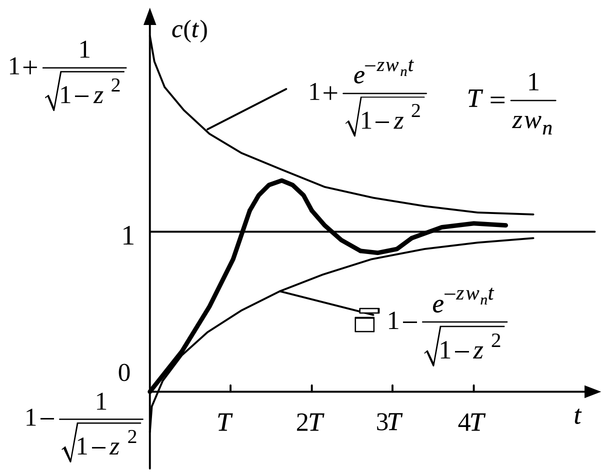
$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$= 1 - \frac{1}{\sqrt{1-z^2}} e^{-zw_n t}\left(\sqrt{1-z^2}\cos w_d t + z \sin w_d t\right)$$

$$= 1 - \frac{e^{-zw_n t}}{\sqrt{1-z^2}} \sin(w_d t + q)$$

$$\theta = \arccos\zeta$$

37



38

2. Critically damped response $z = 1$

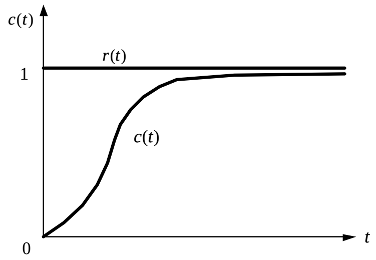
$$s_{1,2} = -zw_n \pm w_n\sqrt{z^2 - 1}$$

$$s_{1,2} = -w_n$$

u(t)

$$c(s) = \frac{w_n^2}{(s^2 + 2w_n s + w_n^2)} \cdot \frac{1}{s} = \frac{1}{s} - \frac{w_n}{(s+w_n)^2} - \frac{1}{s+w_n}$$

$$c(t) = 1 - e^{-zw_n t}(1 + w_n t)$$



39

3. Over damped case $z > 1$

$$s_{1,2} = -zw_n \pm w_n\sqrt{z^2 - 1}$$

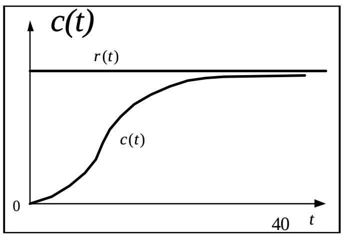
$$c(s) = \frac{1}{s} + \frac{[2(z^2 - z\sqrt{z^2-1} - 1)]^{-1}}{s + zw_n - w_n\sqrt{z^2-1}} + \frac{[2(z^2 + z\sqrt{z^2-1} - 1]^{-1}}{s + zw_n + w_n\sqrt{z^2-1}}$$

Let

$$T_1 = -(z + \sqrt{z^2-1})w_n$$
$$T_2 = -(z - \sqrt{z^2-1})w_n$$

$$c(s) = \frac{1}{s} + \frac{[2(z^2 - z\sqrt{z^2-1} - 1)]^{-1}}{s - T_2} + \frac{[2(z^2 + z\sqrt{z^2-1} - 1]^{-1}}{s - T_1}$$

$$c(t) = 1 + \frac{1}{2\sqrt{z^2 - 1}}\left(\frac{e^{T_1 t}}{-T_1} - \frac{e^{T_2 t}}{-T_2}\right)$$



40

4. undamped case $(z = 0)$

$$s_{1,2} = -zw_n \pm w_n\sqrt{z^2 - 1}$$

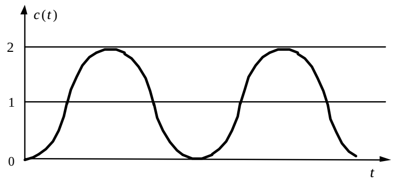
$$s_{1,2} = \pm jw_n$$

$$c(s) = \frac{w_n^2}{(s^2 + w_n^2)} \cdot \frac{1}{s} = \frac{1}{s} - \frac{s}{s^2 + w_n^2}$$

$(z = 0)$

$$c(t) = 1 - \cos w_n t$$

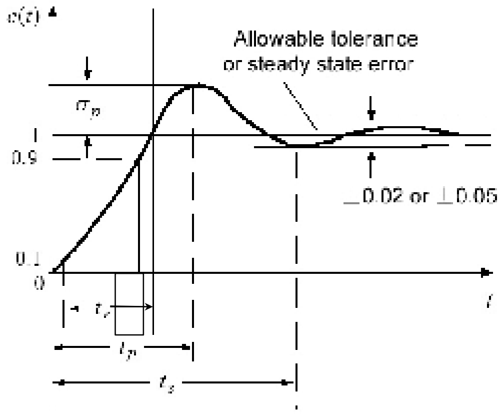
$(t ³ 0)$



41

4.3 Time domain performance specifications

1. Rise time t_r

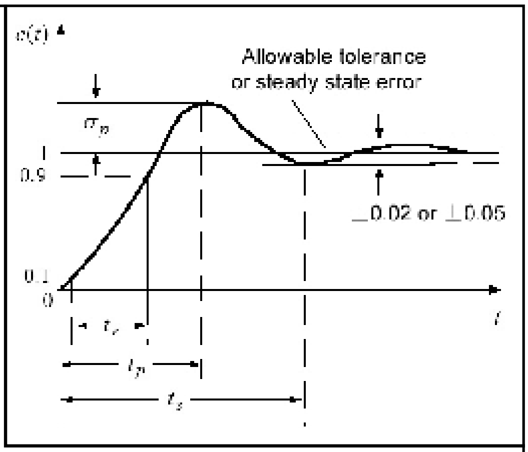


2. The peak time t_p

42

3. Percentage overshoot σ_p

$$\sigma_P = \frac{c(t_P) - c(\infty)}{c(\infty)} \times 100\%$$

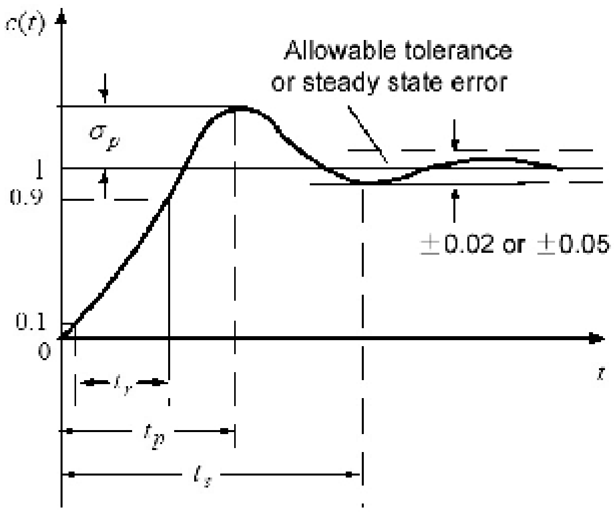


4. Steady-state error e_{ss}

$$e_{ss} = \frac{c(\infty) - r(\infty)}{r(\infty)} \times 100\%$$

43

5. Settling time t_s



44

4.4 The relationship between the performance specifications and system parameters $(0 < z < 1)$

1. Rise time-system parameter

$$c(t) = 1 - \frac{e^{-z w_n t}}{\sqrt{1 - z^2}} \sin(w_d t + q)$$

According to definition: $c(t_r) = 1$

$$c(t_r) = 1 - \frac{e^{-z w_n t_r}}{\sqrt{1 - z^2}} \sin(w_d t_r + q) = 1$$

$$w_d t_r + q = p$$

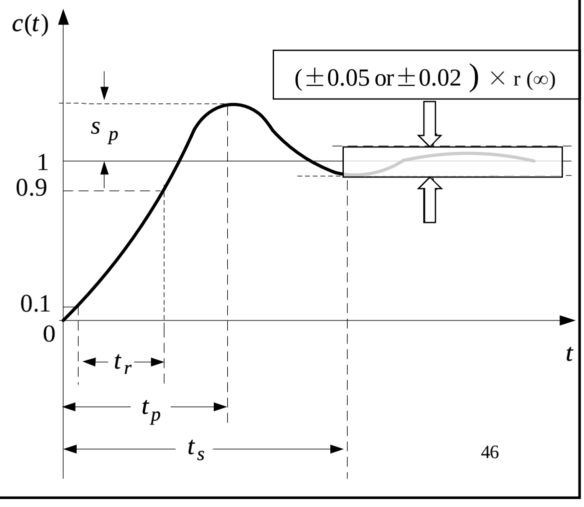
$$t_r = \frac{p - q}{w_d} = \frac{p - q}{w_n \sqrt{1 - z^2}}$$

45

2. Peak time -system parameter

$$2t_p = \frac{2p}{w_d} = \frac{2p}{w_n \sqrt{1 - z^2}}$$

$$t_p = \frac{p}{w_d} = \frac{p}{w_n \sqrt{1 - z^2}}$$



46

3. Percentage overshoot - system parameter

$$c(t) = 1 - \frac{1}{\sqrt{1-z^2}} e^{-zw_n t} (\sqrt{1-z^2} \cos w_d t + z \sin w_d t)$$

$$t_P = \frac{p}{w_d} \qquad w_n t_P = w_n \frac{p}{w_d} = w_n \frac{p}{w_n \sqrt{1-x^2}} = \frac{p}{\sqrt{1-x^2}}$$

$$c(t_p) = 1 - \frac{1}{\sqrt{1-z^2}} e^{-z\, w_n t_p} (\sqrt{1-z^2} \cos w_d t_p + z \sin w_d t_p)$$

$$c(t_P) = 1 - e^{-\frac{z\ p}{\sqrt{1-z^2}}} (\cos p + \frac{z}{\sqrt{1-z^2}} \sin p)$$

47

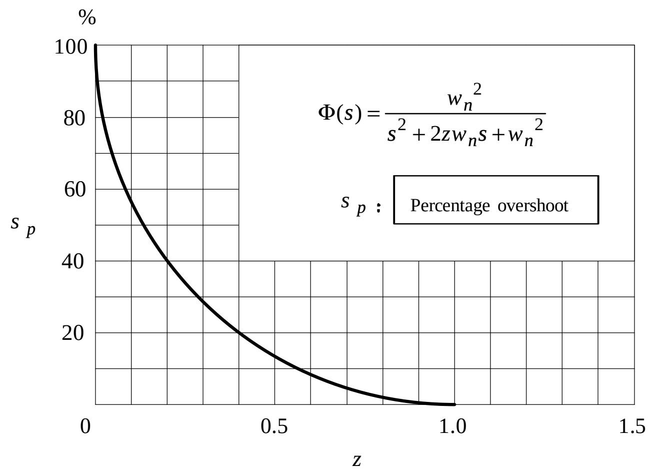
$$c(t_P) = 1 - e^{-\frac{zp}{\sqrt{1-z^2}}} (\cos p + \frac{z}{\sqrt{1-z^2}} \sin p) = 1 + e^{-\frac{zp}{\sqrt{1-z^2}}}$$

$$s_P = \frac{c(t_P) - c(\infty)}{c(\infty)} \times 100\% = \frac{c(t_P) - 1}{1} \times 100\%$$

$$s_P = e^{-\frac{zp}{\sqrt{1-z^2}}}$$

48

$$\sigma_p = e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}}$$



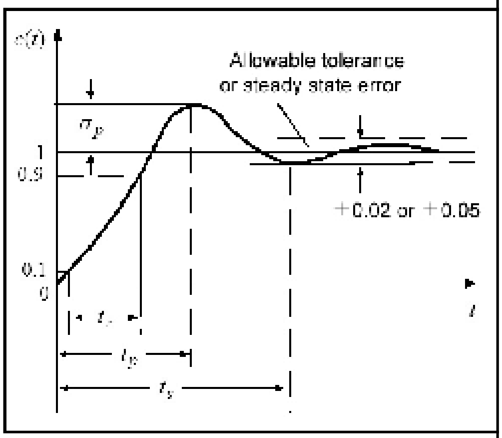
σ_p is only related to ζ.

49

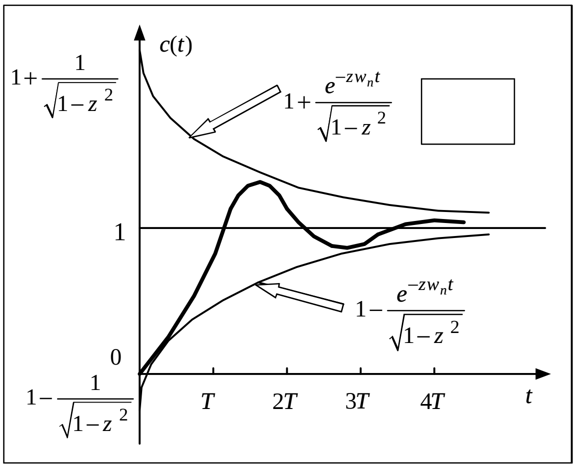
4. The settling time t_s- system parameter

$$\left|c(t_s) - c(\infty)\right| = \Delta \cdot c(\infty)$$

$$\Delta = \pm 0.02 \ or \ 0.05$$



50



$$\left|\frac{e^{-zw_n t_s}}{\sqrt{1-z^2}}\right| = D$$

$$\frac{e^{-zw_n t_s}}{\sqrt{1-z^2}} = \Delta$$

$$t_s = \frac{1}{zw_n} \ln \frac{1}{D\sqrt{1-z^2}}$$

51

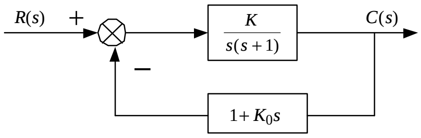
$0 < z < 0.9$ approximately:

$$t_s = \frac{4}{zw_n} \quad \text{for} \quad D = 0.02$$

$$t_s = \frac{3}{zw_n} \quad \text{for} \quad D = 0.05$$

52

Example:



The desired specifications are: $s_P = 20\%$, $t_P = 1s$

What should the value of K and K_0 be? Determine the value of t_s and t_r.

Solution: The transfer function of the closed loop is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)+K+KK_0 s} = \frac{K}{s^2+(1+KK_0)s+K}$$

$$w_n = \sqrt{K}, \quad 2zw_n = 1+KK_0$$

53

(1) z

$$e^{\frac{-zp}{\sqrt{1-z^2}}} = 0.2$$

$$z = 0.456$$

(2) ω_n

$$t_P = \frac{p}{w_d} = \frac{p}{w_n\sqrt{1-z^2}} = 1$$

$$w_n = 3.53 rad/s$$

$$K = w_n^2 = 12.5$$

$$1+KK_0 = 2zw_n$$

$$K_0 = 0.178$$

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)+K+KK_0 s} = \frac{K}{s^2+(1+KK_0)s+K}$$

54

(3) t_r

$$t_r = \frac{p - q}{w_n\sqrt{1 - z^2}} = 0.65s$$

$$q = tg^{-1}\frac{\sqrt{1 - z^2}}{z} = 1.1\ rad$$

(4) t_s

$$t_s = \frac{3}{zw_n} = 1.86s$$ for Δ=5%

55

4.5 Transient response of higher-order systems

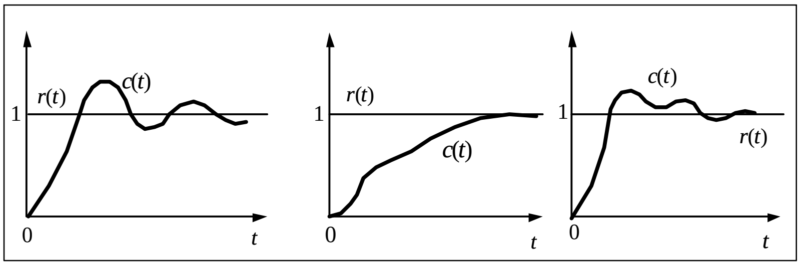
1. Step response of a higher order system

$$C(s) = \frac{k\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}[s+(s_k+jw_k)][(s+(s_k-jw_k)]}\cdot\frac{1}{s}$$

$$C(s) = \frac{a_0}{s} + \sum_{j=1}^{q}\frac{a_j}{s+p_j} + \sum_{k=1}^{r}\frac{a_k s + b_k}{(s+z_k w_k)^2 + \left(w_k\sqrt{1-z_k^2}\right)^2}$$

56

$$C(t) = a_0 + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} e^{-z_k w_k t}(B_k \cos w_k \sqrt{1-z_k^2}t + C_k \sin w_k \sqrt{1-z_k^2}t)$$
$$= a_0 + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{i=1}^{r} D_k e^{-z_k w_k t} \sin(w_k \sqrt{1-z_k^2}\, t + f_k)$$

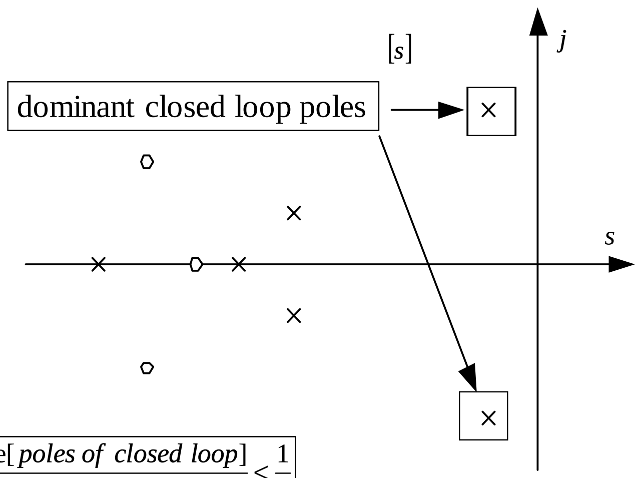


57

3. The dominant poles of higher-order systems

$$C(t) = a_0 + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} e^{-z_k w_k t}(B_k \cos w_k \sqrt{1-z_k^2}t + C_k \sin w_k \sqrt{1-z_k^2}t)$$
$$= a_0 + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{i=1}^{r} D_k e^{-z_k w_k t} \sin(w_k \sqrt{1-z_k^2}\, t + f_k)$$

58



$$\frac{\mathrm{Re}[\textit{poles of closed loop}]}{\mathrm{Re}[\textit{dominant poles}]} < \frac{1}{5}$$

there is no zeros of the closed loop near the dominant poles

59