Chapter 2 Model of control systems

1. Differential equation of control systems
   (from physical and chemical principle)

2. Transfer function (definition and how to get)

3. Block diagram

4. Signal flow chart

1

Transfer function:

$$G(s) = \frac{C(s)}{R(s)}$$

where

$C(s) = L[c(t)]$ C(t) is system output

$R(s) = L[r(t)]$ r(t) is system input

2

Different forms of transfer function:

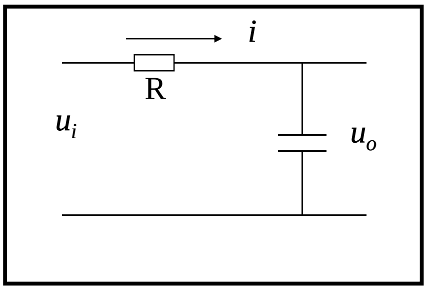
$$G(s)=\frac{b_m s^m + b_{m-1}s^{m-1} + \mathbf{L} + b_1 s + b_0}{a_n + a_{n-1}s^{n-1} + \mathbf{L} + a_1 s + a_0}$$

$$G(s)=\frac{K\prod_{i=1}^{m_1}(t_i s+1)\prod_{k=1}^{m_2}[(\frac{s}{w_k})^2+2x_k(\frac{s}{w_k})+1]}{s^n\prod_{j=1}^{n_1}(T_j s+1)\prod_{l=1}^{n_2}[(\frac{s}{w_l})^2+2x_l(\frac{s}{w_l})+1]}$$

$$G(s)=\frac{K_r(s+z_1)(s+z_2)\,\mathbf{L}\,(s+z_m)}{s^n(s+p_1)(s+p_2)\,\mathbf{L}\,(s+p_q)}=\frac{\prod_{i=1}^{m}(s+z_i)}{s^n\prod_{j=v+1}^{q}(s+p_j)}$$

3

Example:



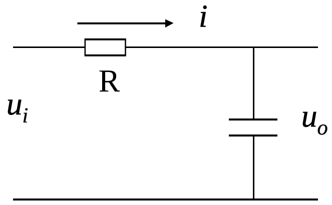
$$Ri(t)+\frac{1}{C}\int i(t)dt=u_i(t)$$

$$u_o(t)=\frac{1}{C}\int i(t)dt$$

4

Eliminating *i(t)*, get

$$RC\frac{du_o(t)}{dt}+u_o(t)=u_i(t)$$



make Laplace transform:

$$(RCs+1)U_o(s)=U_i(s)$$

$$\frac{U_o(s)}{U_i(s)}=\frac{1}{RCs+1}$$

5

Typical factors in transfer functions:

1. Amplifying factor:

$$G(s)=K$$

2. Inertial factor :

$$G(s)=\frac{1}{Ts+1}$$

6

3. differential factor

$$G(s) = Ks$$

4. First order numerator term

$$G(s) = ts + 1$$

5. Second order numerator term

$$G(s) = t^2 s + 2xts + 1$$

7

6. integral factor

$$G(s) = \frac{1}{s}$$

7. oscillatory factor

$$G(s) = \frac{w_n}{s^2 + 2zw_n s + w_n}$$

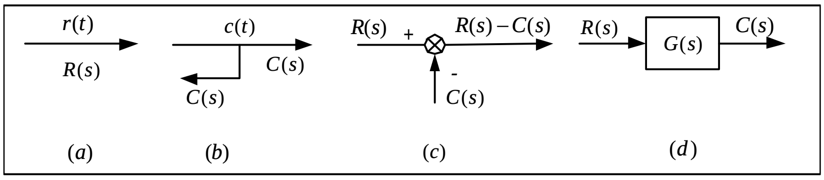
8. Time delay term

$$G(s) = e^{-ts}$$

8

Block diagrams

Basic Elements in block diagrams:



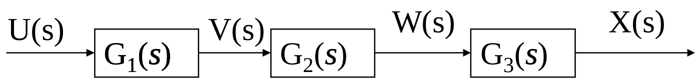
（a）signal line；
（b）pickoff point；
（c）summing junction；
（d）block；

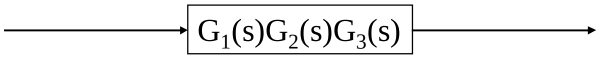
9

- Serial configuration
- General feedback configuration

10

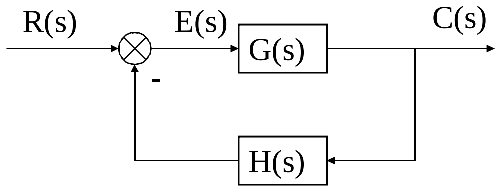
Serial configuration:





11

General feedback configuration:



$$E(s) = R(s) - H(s)C(s)$$

$$C(s) = G(s)\,E(s)$$

hence

$$C(s) = G(s)E(s) = G(s)\,[\,R(s) - C(s)H(s)\,] = G(s)R(s) - G(s)C(s)H(s)$$

$$C(s)[1+G(s)H(s)] = G(s)R(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

12

Simplification of block diagram

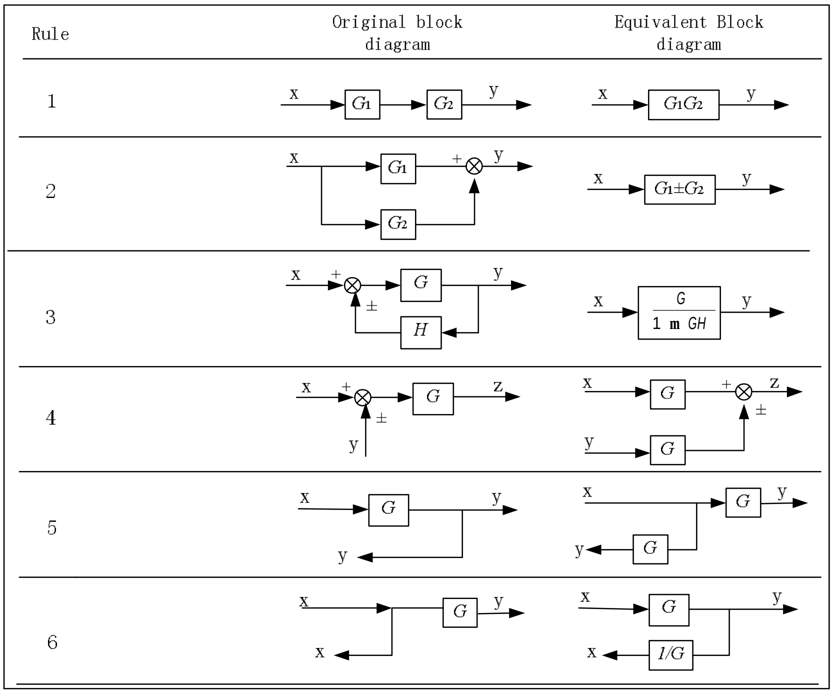
Block manipulation rules

in the process of

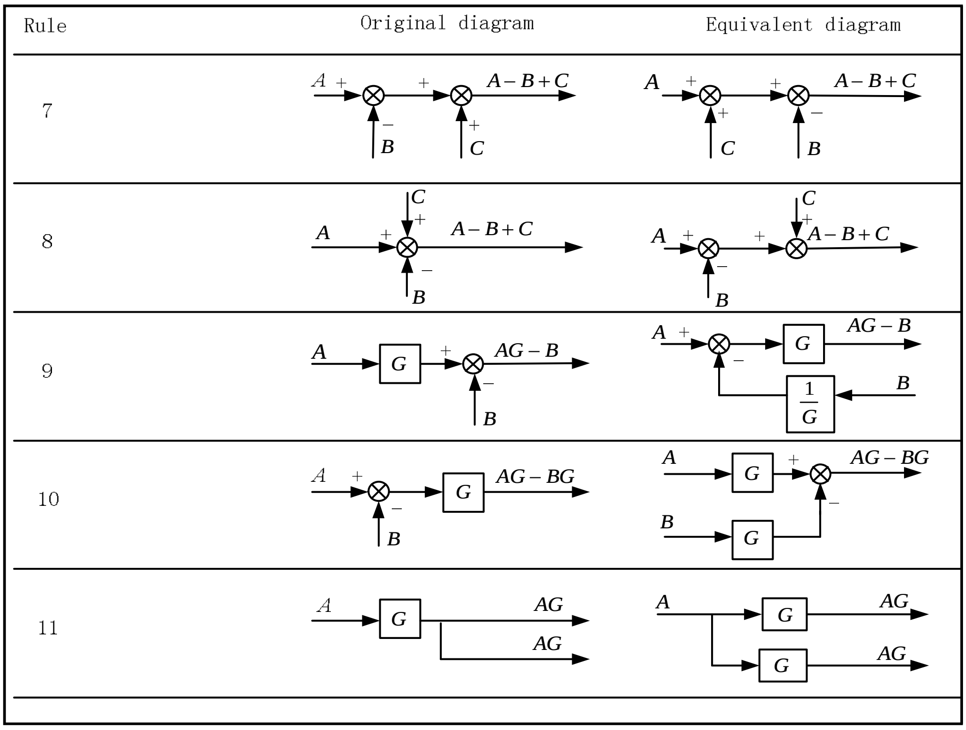
simplifying a block diagram

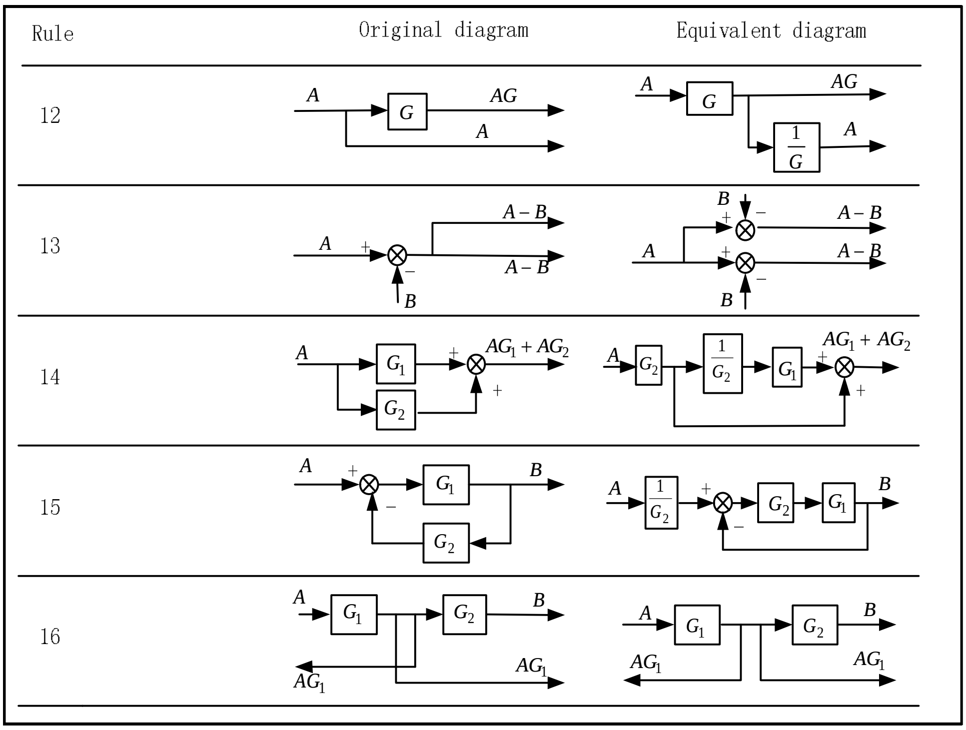
13

Block manipulation rules



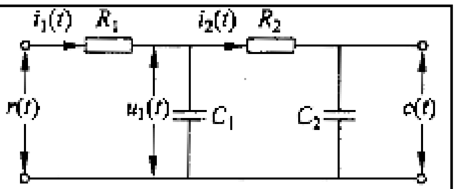
14


Rule
Original diagram
Equivalent diagram
7
A − B + C
8
A − B + C
9
AG − B
1/G
10
AG − BG
11
AG

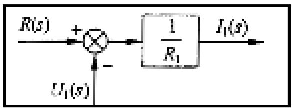

Rule
Original diagram
Equivalent diagram
12
AG
1/G
13
A − B
14
AG1 + AG2
1/G2
15
1/G2
16
AG1

Example:

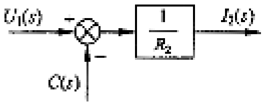
From the principles of the circuits:



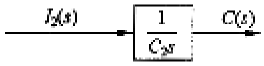
$$\frac{r(t)-u_1(t)}{R_1}=i_1(t)$$



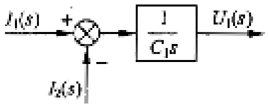
$$\frac{u_1(t)-c(t)}{R_2}=i_2(t)$$



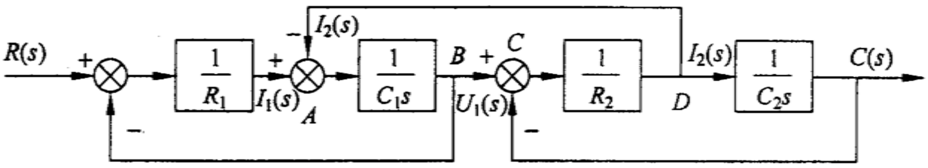
$$c(t)=\frac{1}{C_2}\int i_2(t)dt$$

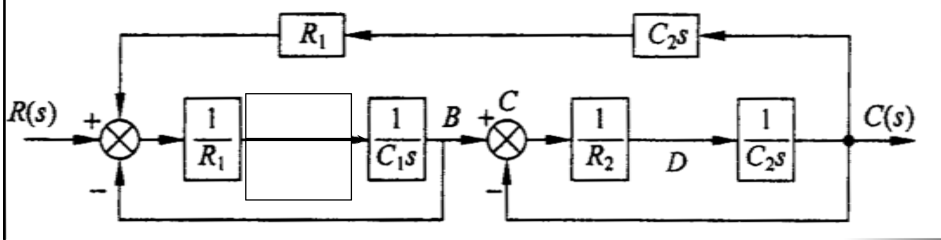


$$u_1(t)=\frac{1}{C_1}\int[i_1(t)-i_2(t)]dt$$

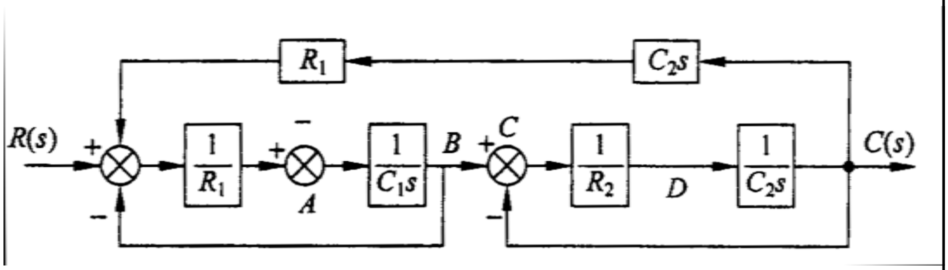


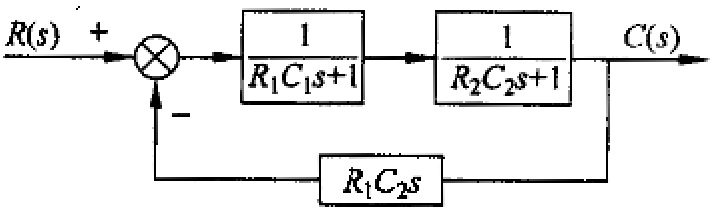
17





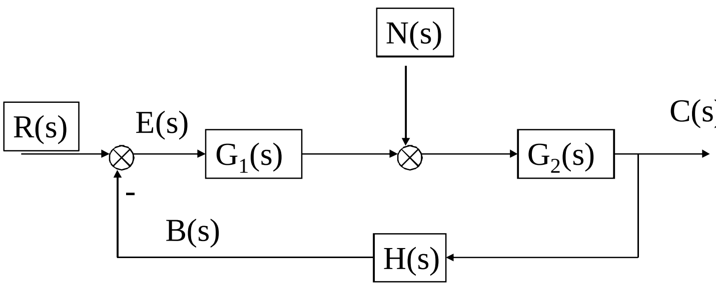
18



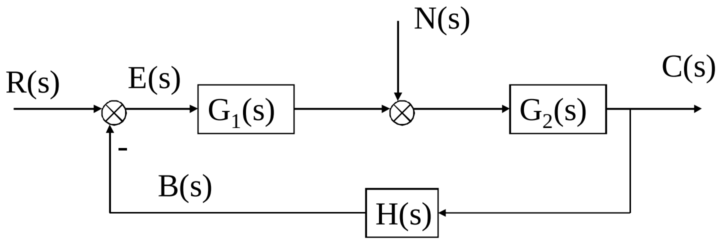


19

Transfer functions when multiple inputs



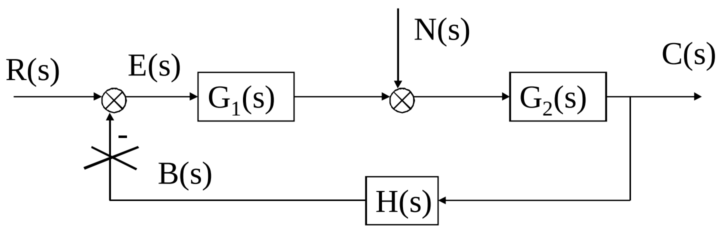
20



$$\frac{C(s)}{E(s)} = G_1(s) \cdot G_2(s) = G(s)$$

$$\frac{B(s)}{C(s)} = H(s)$$

21

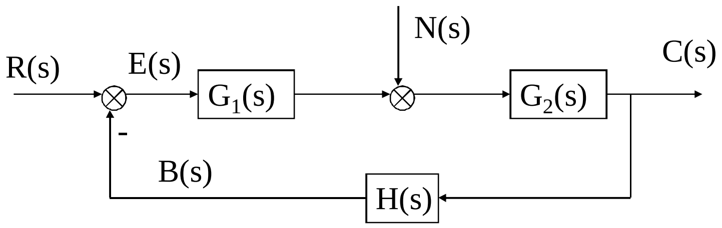


$$\frac{B(s)}{E(s)} = G_1(s) \cdot G_2(s) \cdot H(s) = G(s)H(s)$$

$$\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{C(s)}{N(s)} = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)} = \frac{G_2(s)}{1+G(s)H(s)}$$

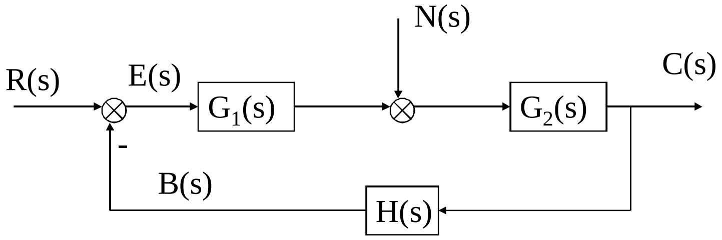
22



$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$\frac{E(s)}{N(s)} = \frac{-G_2(s)H(s)}{1+G(s)H(s)}$$

23



$$C(s) = \frac{G(s)}{1+G(s)H(s)}R(s) + \frac{G_2(s)}{1+G(s)H(s)}N(s)$$

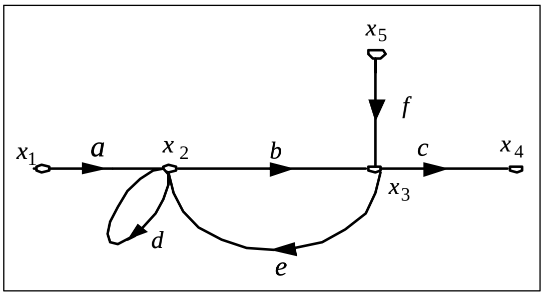
$$E(s) = \frac{1}{1+G(s)H(s)}R(s) - \frac{G_2(s)H(s)}{1+G(s)H(s)}N(s)$$

24

Signal Flow Chart

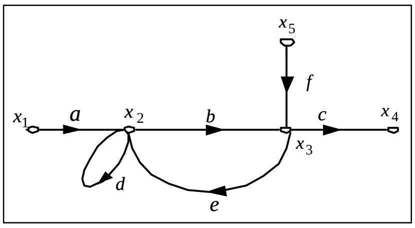
$$\begin{cases} x_1 = x_1 \\ x_2 = ax_1 + dx_2 + ex_3 \\ x_3 = bx_2 + fx_5 \\ x_4 = cx_3 \\ x_5 = x_5 \end{cases}$$

The signal flow chart :



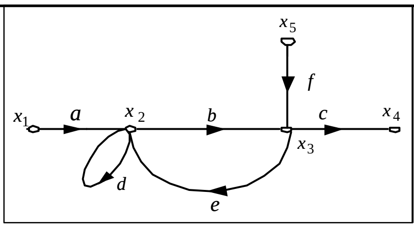
25

Concepts in signal flow chart：



Input node : x_1， x_5

Output node : x_4



Mixed nodes :

x_2, x_3

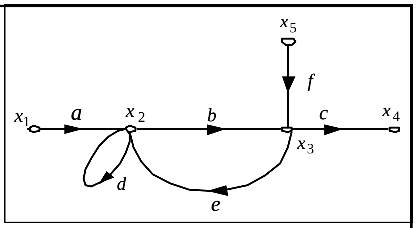
A path:

$x_1 \rightarrow x_2 \rightarrow x_3$,

$x_2 \rightarrow x_3 \rightarrow x_2$

27

A forward path:



$x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4$

A loop:

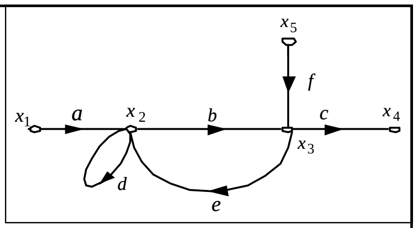
$x_2 \rightarrow x_3 \rightarrow x_2$

Path gain:

Path gain of the path $x_1 \rightarrow x_2 \rightarrow x_3$ is ab

28

Concepts in signal flow chart：



Forward-path gain:

The forward path ($x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4$) gain in the example is abc.

Loop gain:

The loop gain of the loop $x_2 \rightarrow x_3 \rightarrow x_2$ is *be*

Nontouching loops:

29

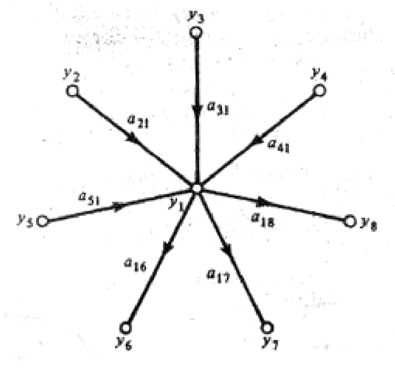
The manipulation of signal flow chart:

$$y_1 = a_{21}y_2 + a_{31}y_3 + a_{41}y_4 + a_{51}y_5$$

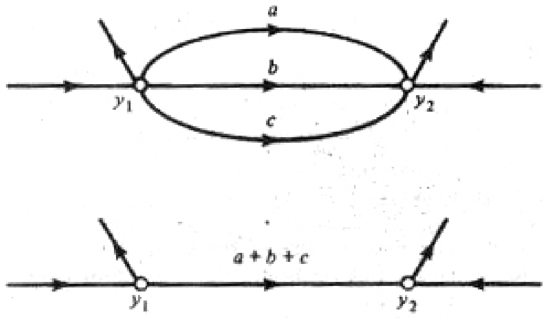
$$y_6 = a_{16}y_1$$

$$y_7 = a_{17}y_1$$

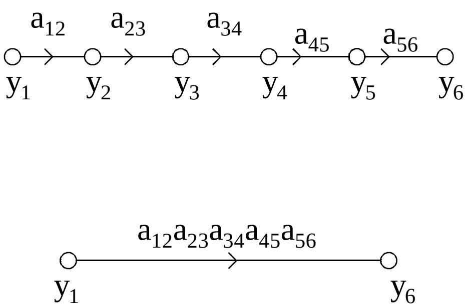
$$y_8 = a_{18}y_1$$



30


a
b
c
y_1
y_2
a + b + c
y_1
y_2

31


a_{12}
a_{23}
a_{34}
a_{45}
a_{56}
y_1
y_2
y_3
y_4
y_5
y_6
$a_{12}a_{23}a_{34}a_{45}a_{56}$
y_1
y_6

32

The Mason Gain Formula

$$G = y_{out} / y_{in} = \frac{1}{\Delta} \sum_{i=1}^{L} p_i \Delta_i$$

where

y_{in} = input node variable, y_{out} = output node variable
G = gain between y_{in} and y_{out}, L = total number of forward paths
P_i = gain of the ith forward path

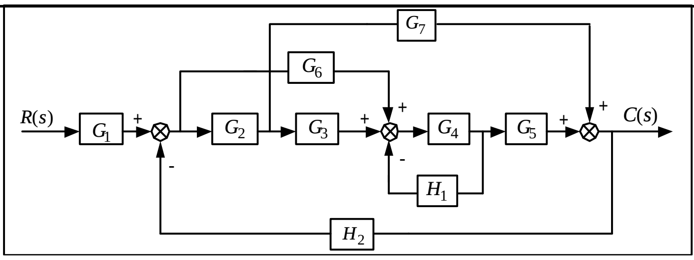
$$\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \mathbf{L}$$

P_{mr} = gain product of the mth possible combination of r nontouching loops

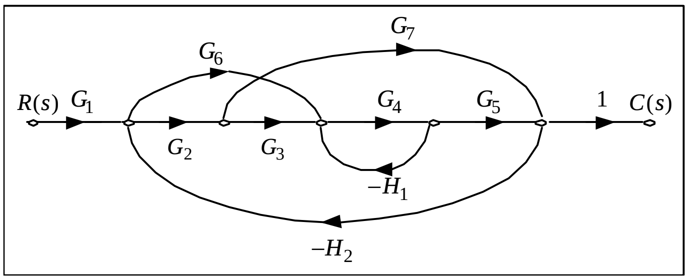
$\mathbf{\Delta_i}$ **=** the **Δ** for that part of the signal flow chart which is nontouching with the ith forward path.

33

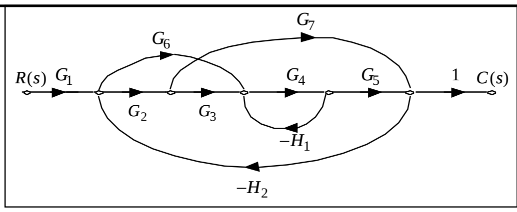
Example:



Signal flow chart:



34



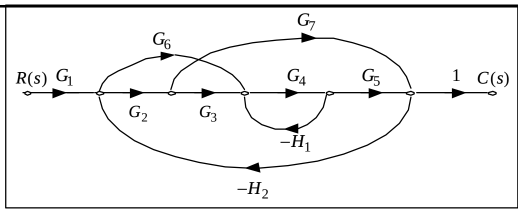
individual loops:

$L_1 = -G_4H_1$ $L_2 = -G_2G_7H_2$

$L_3 = -G_6G_4G_5H_2$ $L_4 = -G_2G_3G_4G_5H_2$

The two nontouching loop is L_1 L_2

35



forward paths:

$P_1= G_1G_2G_3G_4G_5$ $\Delta_1=1$

$P_2= G_1G_6G_4G_5$ $\Delta_2=1$

$P_3= G_1G_2G_7$ $\Delta_3=1-L_1$

36

$L_1 = -G_4H_1$ $L_2 = -G_2G_7H_2$

$L_3 = -G_6G_4G_5H_2$

$L_4 = -G_2G_3G_4G_5H_2$

$P_1 = G_1G_2G_3G_4G_5$ $\Delta_1 = 1$

$P_2 = G_1L_6G_4G_5$ $\Delta_2 = 1$

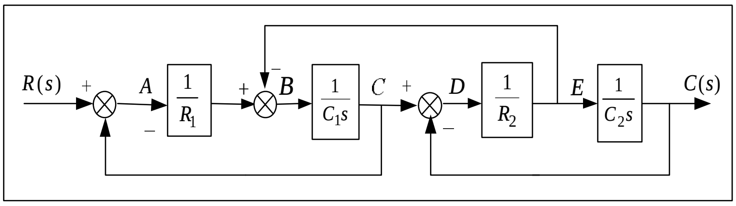
$P_3 = G_1G_2G_7$ $\Delta_3 = 1 - L_1$

$$\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \mathbf{L}$$
$$= 1 - (L_1 + L_2 + L_3 + L_4) + L_1 L_2$$

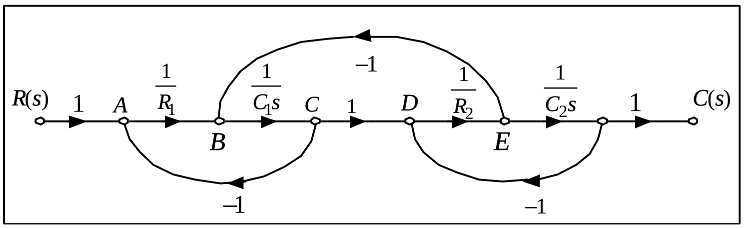
$$\frac{C(s)}{R(s)} = G(s) = \frac{1}{\Delta}(p_1\Delta_1 + p_2\Delta_2 + p_3\Delta_3)$$
$$= \frac{G_1G_2G_3G_4G_5 + G_1G_6G_4G_5 + G_1G_2G_7(1+G_4H_1)}{1+G_4H_1 + G_2G_7H_2 + G_6G_4G_5H_2 + G_2G_3G_4G_5H_2 + G_4H_1G_2G_7H_2}$$

37

Example2:

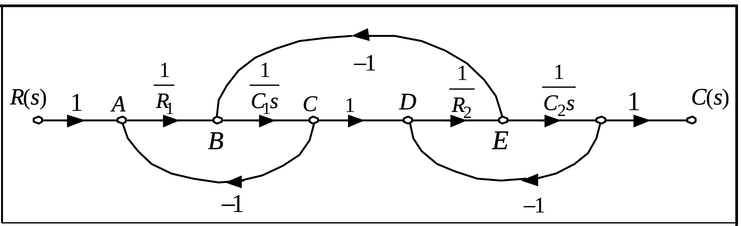


Signal flow chart:



38

Example2:



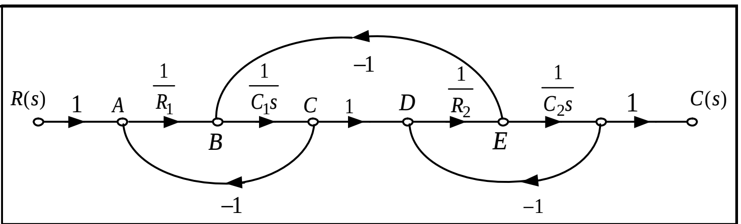
individual loops:

$L_1 = \frac{-1}{R_1C_1s}$ $\quad L_2 = \frac{-1}{R_2C_2s}$ $\quad L_3 = \frac{-1}{R_2C_1s}$

Nontouching loop: $L_1L_2 = \frac{1}{R_1C_1sR_2C_2s^2}$

$$\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \mathbf{L}$$
$$= 1 - (L_1 + L_2 + L_3) + L_1L_2$$
$$= 1 + \frac{1}{R_1C_1s} + \frac{1}{R_2C_2s} + \frac{1}{R_2C_1s} + \frac{1}{R_1C_1R_2C_2s^2}$$

39



forward path:

$P_1 = \frac{1}{R_1R_2C_1C_2s^2}$ $\qquad \Delta_1 = 1$

$$\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \mathbf{L}$$
$$= 1 - (L_1 + L_2 + L_3) + L_1L_2$$
$$= 1 + \frac{1}{R_1C_1s} + \frac{1}{R_2C_2s} + \frac{1}{R_2C_1s} + \frac{1}{R_1C_1R_2C_2s}$$

$$\frac{C(s)}{R(s)} = G = \frac{P_1\Delta_1}{\Delta} = \frac{1}{R_1R_2C_1C_2s^2 + R_2C_2s + R_1C_1s + R_1C_2s + 1}$$

40