Chapter 6: Frequency Domain Anaysis

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Outline

1. Introduction of Frequency Response
2. Frequency Response of the Typical Elements
3. **Nyquist and Bode Diagram of Open-loop System**
4. Nyquist-criterion
5. Frequency Response Based System Analysis
6. Frequency Response of Closed-loop Systems

Concept

Graphical method

Analysis

6.3.4 Minimum Phase System

Definition: A system that has no open-loop zeros or poles in the right hand half plane is called minimum phase system.

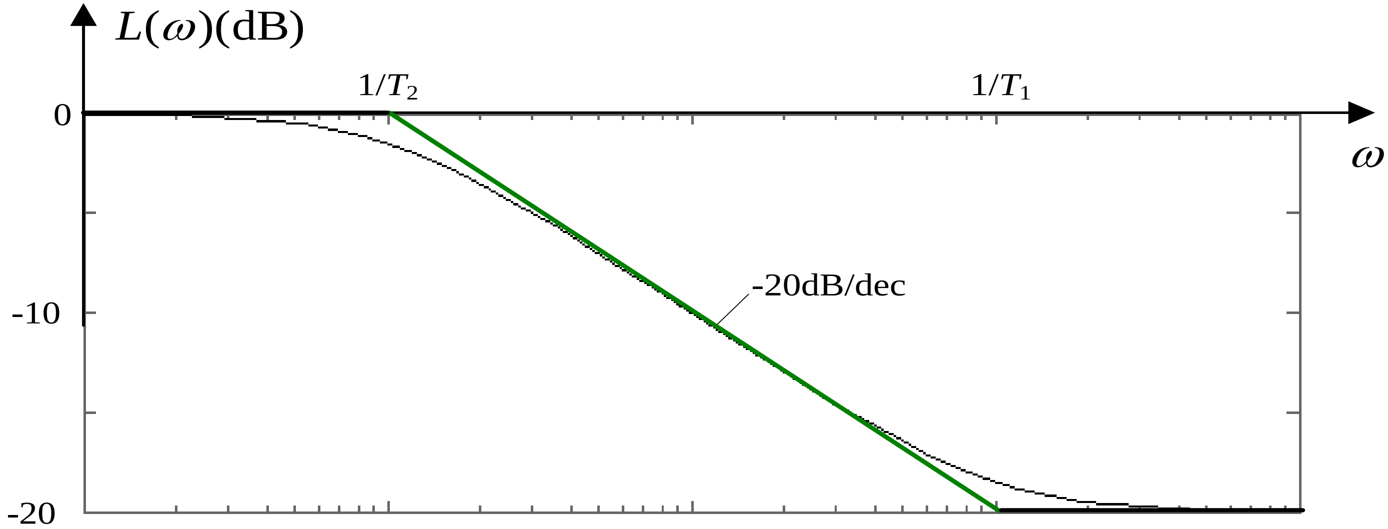
- **Non-minimum phase system: Systems with at least one open-loop pole or zero in the right-hand half s plane are known as non-minimum phase system.**
- **A transfer system with delay element is a non-minimum phase system.**
- **For the systems with same magnitude characteristics, the absolute value of phase characteristics of minimum phase systems are minimum.**
- **For minimum phase systems, the magnitude characteristics are uniquely related to the phase characteristics. For non-minimum phase system, there is no such relation.**

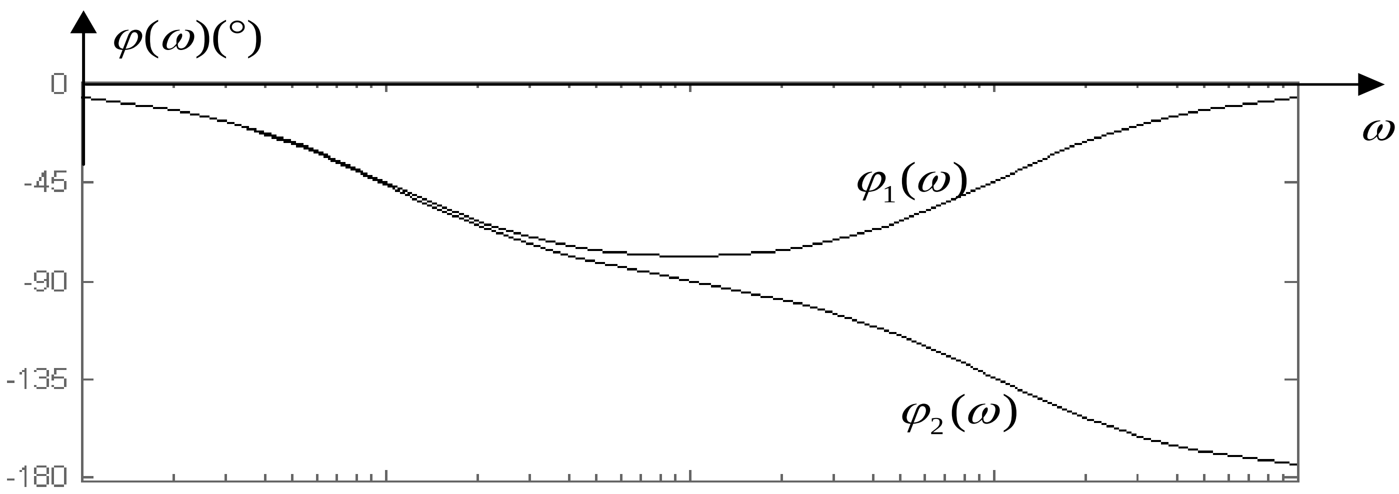
Example 5.8: The frequency characteristics of a minimum phase system is shown as

$$G(j\omega)=\frac{1+jT_1\omega}{1+jT_2\omega} \qquad (T_2>T_1>0)$$

The frequency characteristics of a non-minimum phase system is shown as

$$G(j\omega)=\frac{1-jT_1\omega}{1+jT_2\omega} \qquad (T_2>T_1>0)$$





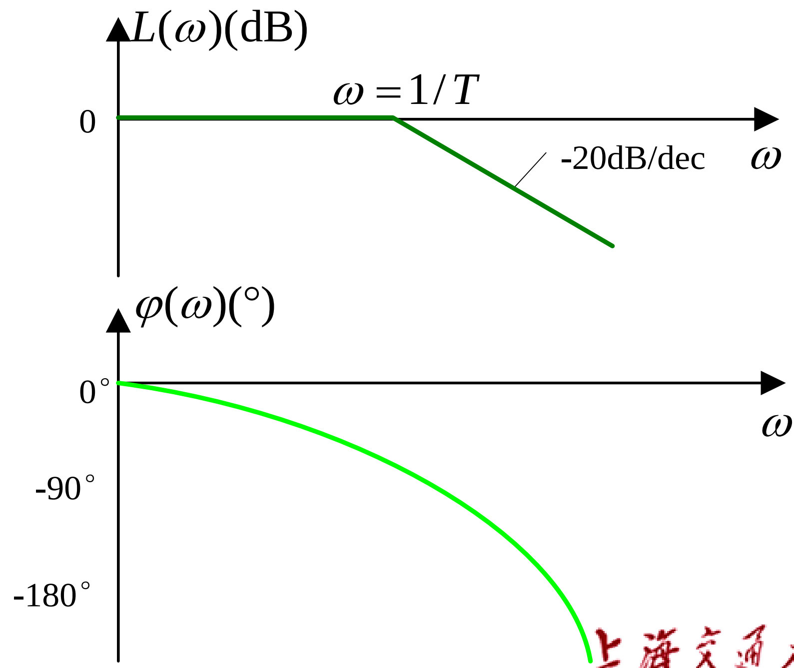
Bode Diagrams for Minimum and Non-minimum Phase System

- **Example 6.9: Sketch the Bode Diagram of the following system**

$$G(s)=\frac{e^{-\tau s}}{Ts+1}$$

Solution: $A(\omega)=\dfrac{1}{\sqrt{1+\omega^2T^2}}$ $\qquad$ $\varphi(\omega)=-\tau\omega-\operatorname{arctg}\omega T$

- **It is seen that the magnitude characteristics is same as the inertial element, but the phase characteristics has the extra item $-\tau\omega$.**
- **It implies the phase is lagged very quickly.**



Example 6.10: **The following five systems have same magnitude characteristics.**

$$G_1(s)=\frac{T_2s+1}{T_1s+1} \qquad \varphi_1(\omega)=tg^{-1}T_2\omega-tg^{-1}T_1\omega$$

$$G_2(s)=\frac{1-T_2s}{T_1s+1} \qquad \varphi_2(\omega)=-tg^{-1}T_2\omega-tg^{-1}T_1\omega$$

$$G_3(s)=\frac{T_2s+1}{1-T_1s} \qquad \varphi_3(\omega)=tg^{-1}T_2\omega+tg^{-1}T_1\omega$$

$$G_4(s)=\frac{1-T_2s}{1-T_1s} \qquad \varphi_4(\omega)=-tg^{-1}T_2\omega+tg^{-1}T_1\omega$$

$$G_5(s)=\frac{T_2s+1}{T_1s+1}e^{-\tau s} \qquad \varphi_5(\omega)=tg^{-1}T_2\omega-tg^{-1}T_1\omega-57.3\times\omega\tau$$

$$A_1(\omega)=A_2(\omega)=A_3(\omega)=A_4(\omega)=A_5(\omega)=\frac{\sqrt{1+(T_2\omega)^2}}{\sqrt{1+(T_1\omega)^2}}$$

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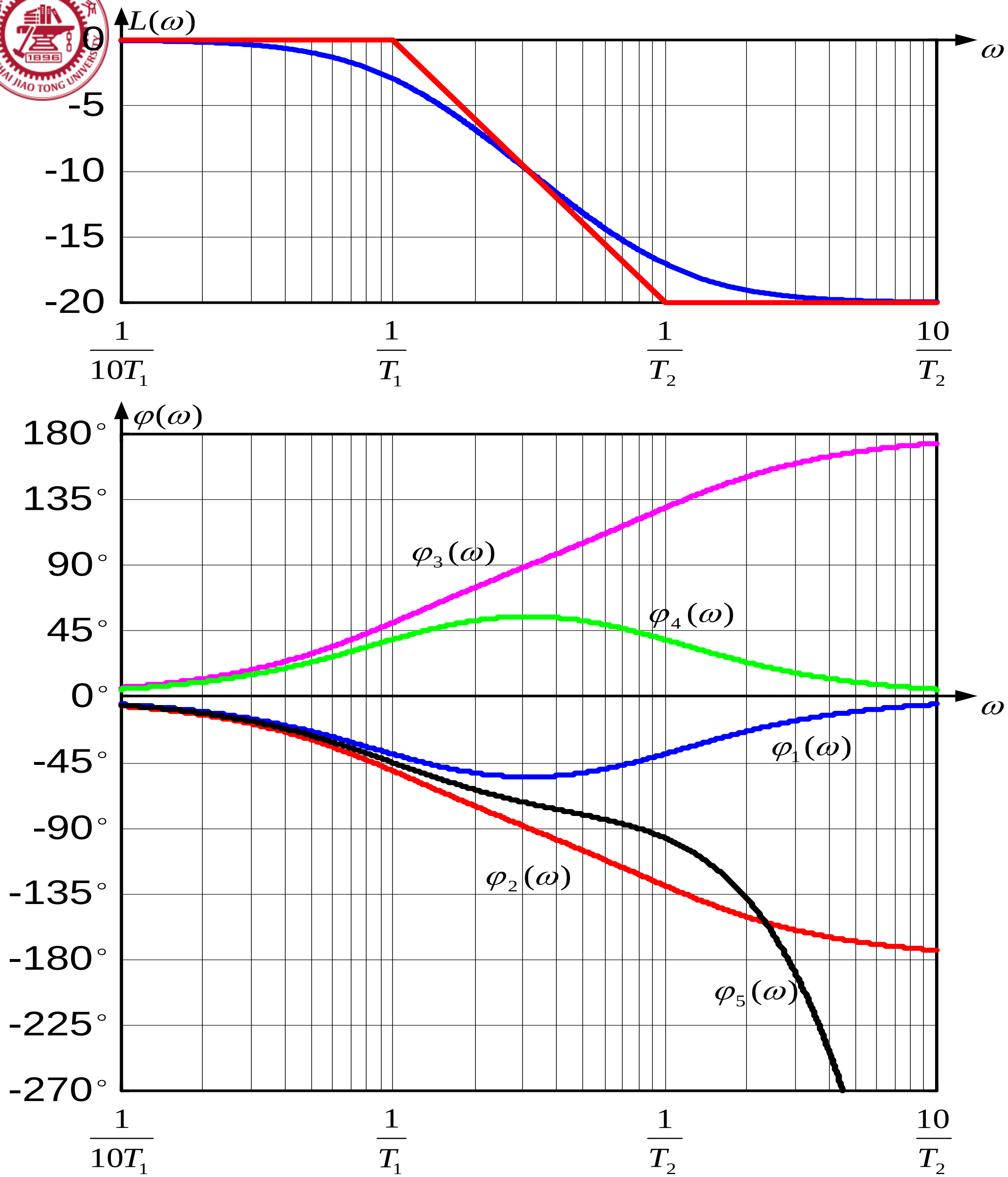
Assume that $T_1 = 10T_2$ **and** $\tau = T_2$ **.**

The point at $\omega = \sqrt{10}/T_1$ **is the geometric intermediate point between** $\frac{1}{T_1}$ **and** $\frac{1}{T_2}$**.**

ω	$1/(10T_1)$	$1/T_1$	$\sqrt{10}/T_1$	$1/T_2$	$10/T_2$
$j_1(w)$	-5.1°	-39.3°	-54.9°	-39.3°	-5.1°
$j_2(w)$	-6.3°	-50.7°	-90°	-129.3°	-173.7°
$j_3(w)$	6.3°	50.7°	90°	129.3°	173.7°
$j_4(w)$	5.1°	39.3°	54.9°	39.3°	5.1°
$j_5(w)$	-5.7°	-45°	-73°	-96.6°	-578.1°



Remarks:

For $\omega=[0,+\infty)$, minimum phase system has the minimum phase change. Moreover, the phase change has the same tendency with magnitude characteristics. ($\varphi_1(\omega)$)

Non-minimum phase system has bigger phase change (eg. $\varphi_2(\omega)$、$\varphi_3(\omega)$、$\varphi_5(\omega)$) .

Even if the phase change is minimum, the phase change tendency is different from that of magnitude change (eg. $\varphi_4(\omega)$).

6-4 Nyquist Stability Criterion

Two kinds of frequency domain stability criteria

Nyquist stability criterion and Bode stability criterion

- **Nyquist stability criterion: based on Nyquist diagram of open-loop system to determine the stability of closed-loop system；**
- **Bode stability criterion: based on Bode diagram of open-loop system to determine the stability of closed-loop system ；**
- **Two method are same in nature**

Key point for frequency domain stability criteria：

- **Determine the stability of closed-loop system based on the frequency response of open-loop system**
- **Moreover, they can be used to determine relative stability.**

- **Advantage of Nyquist stability criterion**
  - **Geometric criterion: simplicity, graphical method, low computational complexity (Routh stability criterion is an algebraic method）**
  - **No need to know the differential equation or transfer function. The frequency responses can be obtained by mathematical modeling or experimental method.**
  - **Helpful to understand the concept of relative stability**
- **Mathematical basis for Nyquist stability criterion:**

  Conformal Mapping (Cauchy’s Theorem)

  保角映射，又称柯西定理

6.4.1 Conformal Mapping

1. Mapping between *s* plane and *F*(*s*) plane

Given a complex function

$$F(s)=1+G(s)H(s)=\frac{K_r(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$s=\sigma+j\omega$，$F(s)$ is a complex function，and let $F(s)=U+jV$

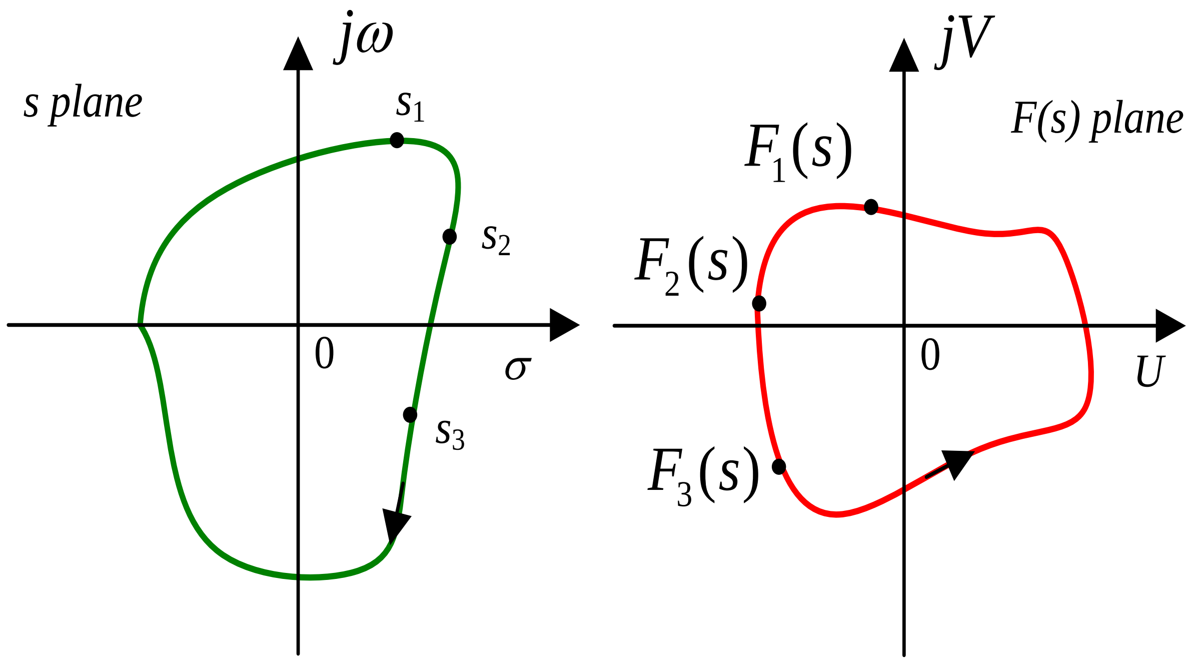
- **Assume *F*(*s*) is an analytic function (解析函数) except with finite singularity (有限奇点).**
- **The number of zeros of *F(s)* is same as that of poles.**

The zeros of *F(s)* are the poles of the closed-loop system.

The poles of *F(s)* are the poles of the open-loop system.

A close contour Γ_s in *s* plane which does not go through any singularity

There exists the mapping contour Γ_F in $F(s)$ plane



Mapping between *s* plane and $F(s)$ plane

Remarks:

① The Γ_F contour in *F(s)* plane may go clockwise or anti-clockwise when Γ_s (s) moves clockwise in *s* plane. It depends on *F(s)*.

② What we are interested in is that Γ_s (s) encircles the origin a total of *N* times and the direction rather than the curve shape.

③ The number *N* and direction are related to the stability of closed-loop system.（corresponding to the phase change of *F*(*s*)）

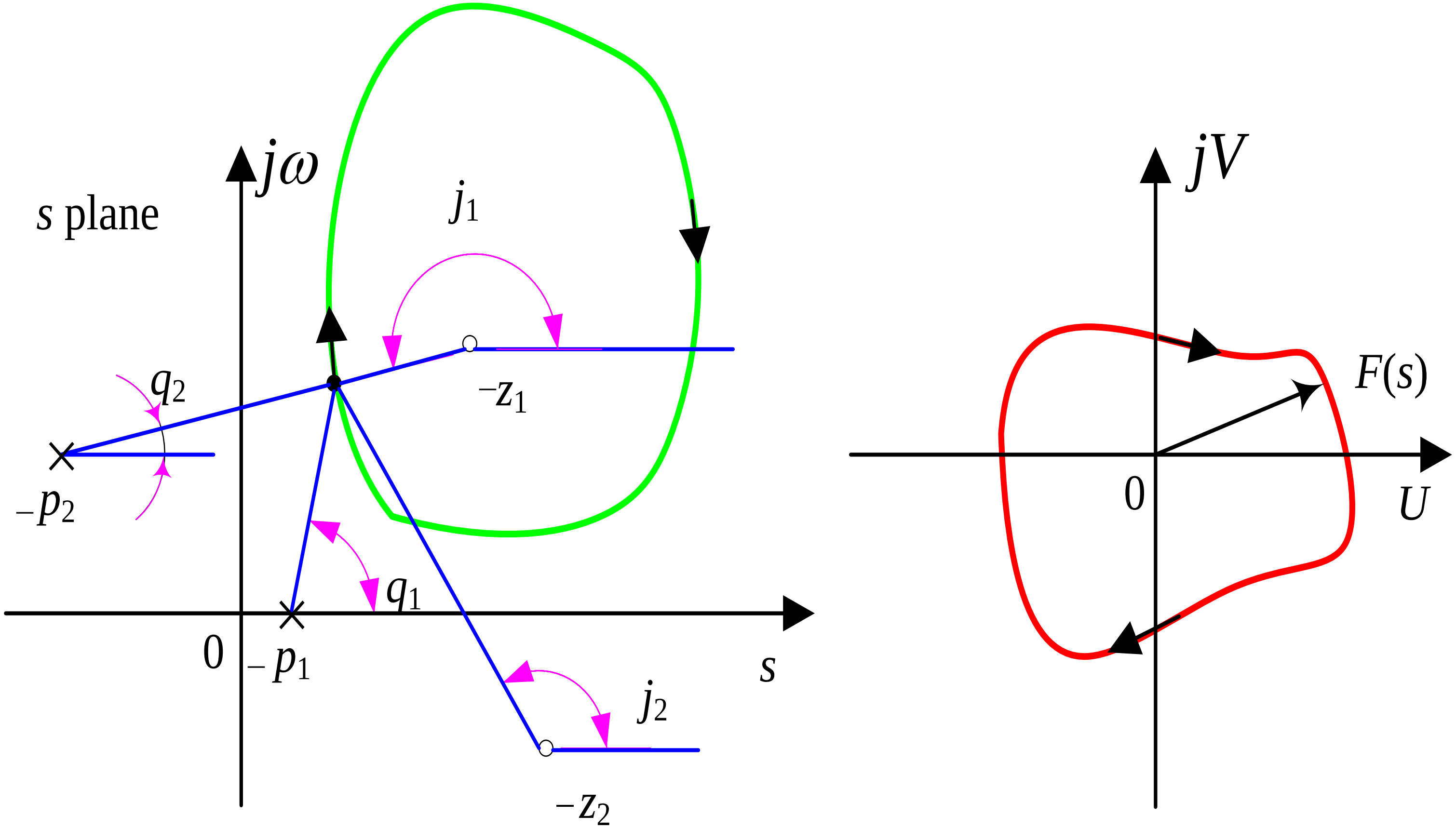
2. Phase of $F(s)$

The phase of $F(s)$ can be obtained that

$$\angle F(s) = \sum_{i=1}^{n} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j)$$

Assumption: The contour Γ_s encircles the zero $-z_1$, and other zeros and poles locate outside of Γ_s.

- When the variable s traverses the contour Γ_s in the clockwise direction, the phase change of vector $(s+z_1)$ is -2π and that of other vectors are zero. The phase change of $F(s)$ is -2π.
- If the contour Γ_s encircles Z zeros, the mapping Γ_F in the plane of $F(s)$ encircles the origin Z times in the clockwise direction.



Contour Γ_s encircle $–z_1$

- Similarly, assume Γ_s encircles P poles. When the variable s traverses the contour Γ_s in the clockwise direction, then Γ_F in the plane of $F(s)$ encircles the origin P times in the anti-clockwise direction.

3. Conformal mapping: For a given contour Γ_s in the s plane that encircles P poles and Z zeros of the function $F(s)$ in a clockwise direction, the resulting contour Γ_F in the $F(s)$ plane encircles the origin a total of N times in a clockwise dirction, where

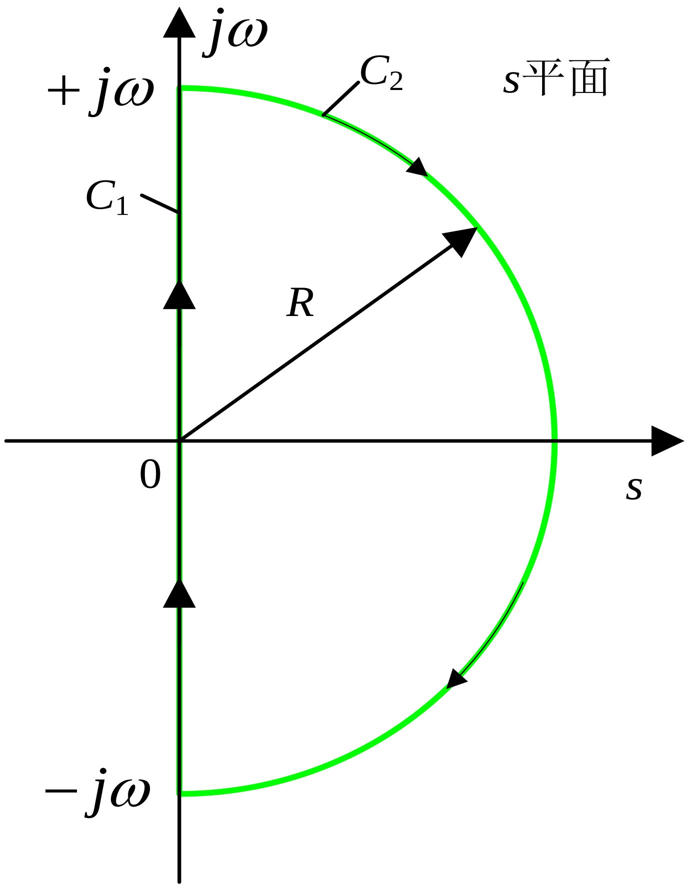
$$N=Z-P$$

How to use the conformal mapping to determine the stability of closed-loop system?

6.4.2 Nyquist Stability

1. Nyquist contour

- The sufficient and necessary condition for stability of closed-loop system is that all the zeros of $F(s)$ locate on the left-hand half open s plane.
- In order to determine that closed-loop stability, we need to check if there is a zero of $F(s)$ locate the whole of the right-hand side of s plane.
- Nyquist contour: encompasses the whole right-hand side of s plane in the clockwise direction. It can encircle all the zeros and poles at the right-hand half side of s plane.
- Include two separate segments:
  - Straight line C_1 along the imaginary axis $s=j\omega$ with $\omega\in(-\infty,+\infty)$
  - Infinite semicircle C_2 with infinite radius centered at the origin



Nyquist contour without open-loop poles on the imaginary axis

2. Conformal mapping

- **Based on conformal mapping, when s traverses along the Nyquist contour for one time, the mapping contour Γ_F in $F(s)$ plane encircles the origin $N = Z - P$ times in the clockwise direction, where Z and P are respectively the number of zeros and poles in the right-hand half s plane.**
- **The sufficient and necessary condition for stability of the closed-loop system is that there is no zeros of $F(s)$, i.e. $Z=0$.**

Analysis:

- **When *s* traverses along the Nyquist contour for one time, and the mapping contour Γ_F in *F*(*s*) plane encircles the origin $N = -P$ times in the clockwise direction, then the closed-loop system is stable.（i.e. *P* times in the anti-clockwise direction）**

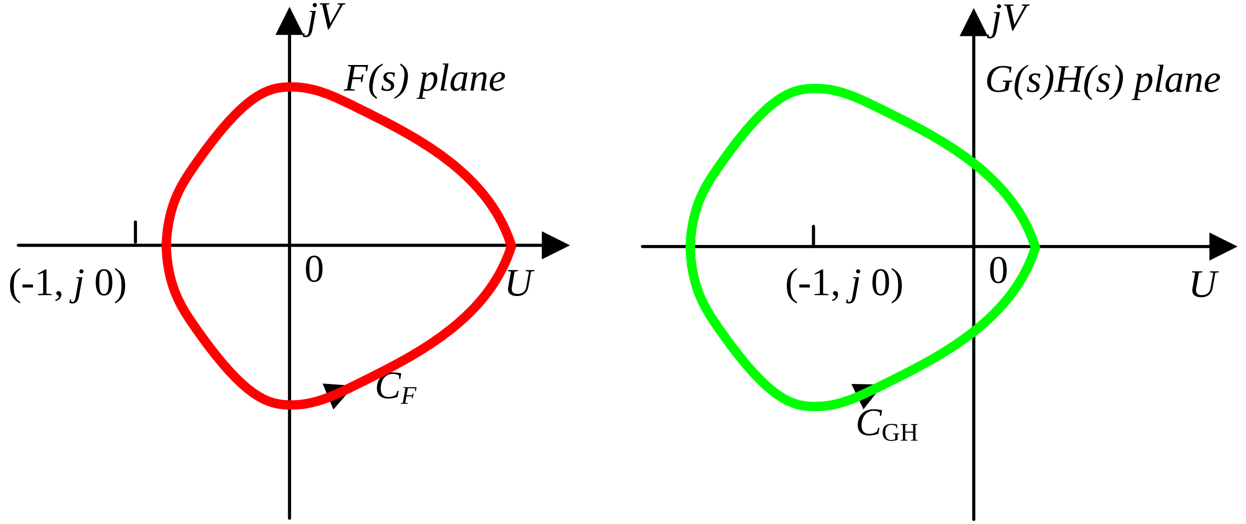
- **If $N \neq -P$, the closed-loop system is unstable. The number of poles in the right-hand half *s* plane of closed-loop system is $Z = N + P$.**

- **If the open-loop system is stable, i.e. $P=0$, then the condition for the stability of closed-loop system is: the mapping contour C_F does not encircle the origin, i.e. $N=0$.**

- **Characteristic equation of the system**

$$G(s)H(s) = F(s) - 1$$

- **When the contour $F(s)$ encircles the origin, the contour Γ_{GH} *of* $G(s)H(s)$ equivalently encircles the point $(-1, j0)$.**



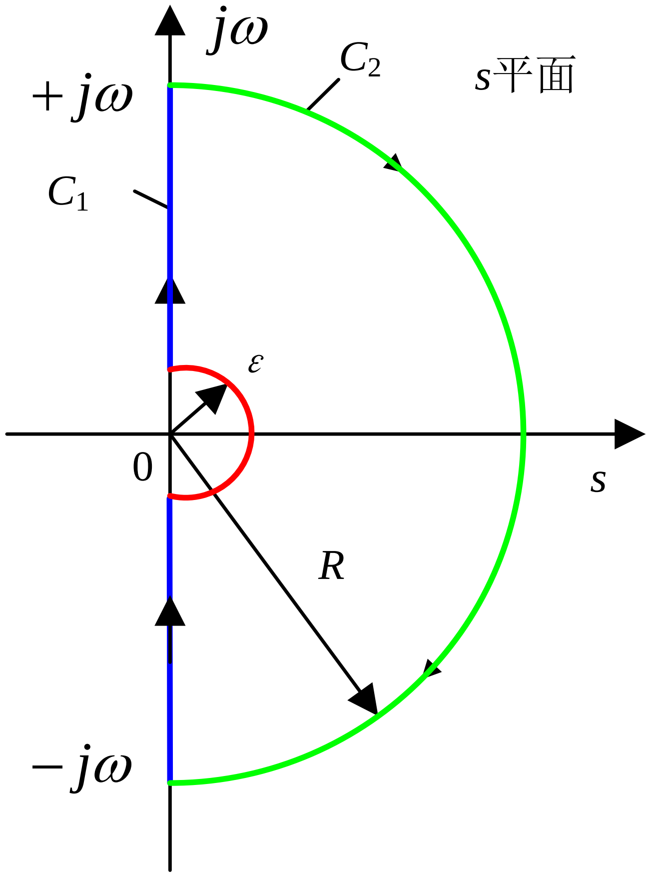
The mapping of Nyquist contour on $F(s)$ plane and $G(s)H(s)$ plane

How to draw the mapping contour Γ_{GH}?

- **Mapping of C_1: Let $s=j\omega$ and substitute into $G(s)H(s)$, and then we obtain open-loop frequency characteristics $G(j\omega)H(j\omega)$. Draw the Nyquist diagram. Draw the symmetrical part over the real axis for $\omega\in(-\infty,0^-]$.**
- **When there are open-loop poles at the origin, the part from s=j0⁻ to s=j0⁺ of the Nyquist contour is replaced by a half circle with very tiny radius centered at the origin, i.e.** $s=\varepsilon\cdot e^{j\theta}$

$$G(s)H(s)\Big|_{s=\varepsilon\cdot e^{j\theta}}=\frac{K\prod_{i=1}^{m}\left(\tau_i\varepsilon e^{j\theta}+1\right)}{\left(\varepsilon e^{j\theta}\right)^{v}\prod_{j=v+1}^{n}\left(T_j\varepsilon e^{j\theta}+1\right)}=\frac{K}{\varepsilon^{v}}e^{-jv\theta}$$

- **Draw the open-loop frequency responses for ω from $-\infty$ to $+\infty$, and we can obtain the whole mapping contour Γ_{GH}.**



Nyquist contour for the open-loop system with poles on the imaginary axis

- **Difference of two kinds of Nyquist contour：**
  - **The second kind of Nyquist contour has the center at origin and very tiny radius ε. *For* $\varepsilon \to 0$, the area of half circle on the right-hand plane tends to zero.**
- **The zeros and poles of $F(s)$ on the right-hand half s plane are encircled by the Nyquist contour. The open-loop poles at the origin is partitioned into the left-hand half plane.**
- **It needs to know the number of open-loop poles on the left-hand and right-hand half planes.**

3. Nyquist stability criterion

The sufficient and necessary condition for stability of closed-loop system is that: for ω from $-\infty$ to $+\infty$, the open-loop frequency response $G(j\omega)H(j\omega)$ encircle the point $(-1, j0)$ for $N=P$ times in the anti-clockwise direction (or $N=-P$ times in the clockwise direction), where P is the number of open-loop poles on the right-hand half s plane.

- **If $N \neq -P$, the closed-loop system is unstable. The number of poles in the right-hand half s plane of closed-loop system is $Z=N+P$.**
- **If the open-loop system is stable, i.e. $P=0$, then the condition for the stability of closed-loop system is: the mapping contour C_{GH} does not encircle the point $(-1, j0)$, i.e. $N=0$.**

- **Example 6.11: Given the open-loop transfer function**

$$G(s)H(s) = \frac{K}{(0.5s+1)(s+1)(2s+1)}$$

Please draw the Nyquist diagram for (1) *K*=5, (2) *K*=15, and determine the stability of the closed-loop system.

4、Nyquist stability criterion for systems with open-loop poles on the imaginary axis

- **Normally with integration elements，i.e. with open-loop poles at the origin of *s* plane**
- **The second kind of Nyquist contour is applicable.**
- **When *s* traverses along the little half circle from $\omega=0^-$ to $\omega=0^+$, θ changes from －90° to 0° and then further to ＋90° (counterclockwise). The mapping on $G(s)H(s)$ plane is a semicircle with infinite radius. The movement is in the direction of clockwise from $v\cdot 90°$ to 0° and further to $-v\cdot 90°$.**

ω: $0^- \to 0^+$;

θ: $-90° \to 0° \to +90°$;

$\varphi(\omega)$: $+v\cdot 90° \to 0° \to -v\cdot 90°$

$$G(s)H(s)\Big|_{s=\varepsilon\cdot e^{j\theta}} = \frac{K\prod_{i=1}^{m}\left(\tau_i \varepsilon e^{j\theta}+1\right)}{\left(\varepsilon e^{j\theta}\right)^{v}\prod_{j=v+1}^{n}\left(T_j \varepsilon e^{j\theta}+1\right)} = \frac{K}{\varepsilon^{v}} e^{-jv\theta}$$

Example 6.12: **Please draw the Nyquist diagram for the open-loop system**

$$G(s)H(s)=\frac{10}{s(s+1)(s+2)}$$

and determine the stability.

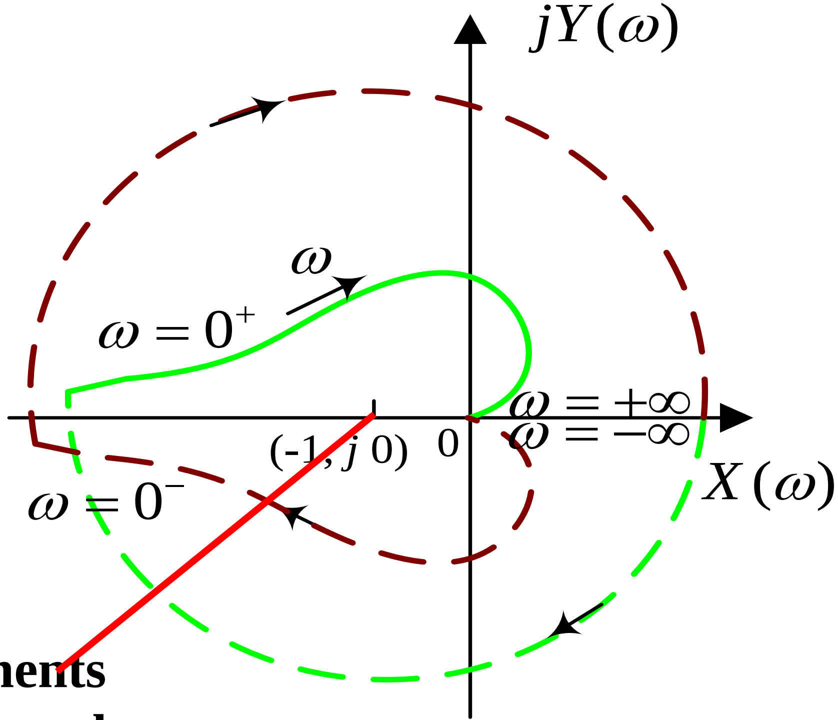
Example 6.13: **Draw the Nyquist diagram for the system with open loop transfer function as follows:**

$$G(s)H(s) = \frac{10}{s^2(s+1)(s+2)}$$

Determine the stability of the closed-loop system.

5. How to calculate N?

In order to calculate the number of encirclement of the critical point, draw a straight line radially outward from the critical point cutting all paths of the Nyquist diagram.



Nyquist Diagram

$$N = N_+ - N_-$$

N_+: **the number of clockwise encirclements**

the number of times the path crosses the line in a clockwise sense

N_-: **the number of counterclockwise encirclements**

the number of times the path crosses the line in the counterclockwise sense

6.4.3 Nyquist Stability Criterion for Bode Diagram

Bode diagram based stability criterion is alternative form of Nyquist stability criterion.

It uses Bode diagram of open-loop system to determine the stability of closed-loop system.

1. Relationship between Nyquist diagram and Bode diagram

- **The unity circle centered at the origin←→ *0*dB line of Bode diagram**
- **Outside of the unity circle←→$L(\omega)>0$**
- **Inside of the unity circle←→$L(\omega)<0$**
- **Negative real axis of Nyquist diagram←→ -180° line of the log-phase diagram**

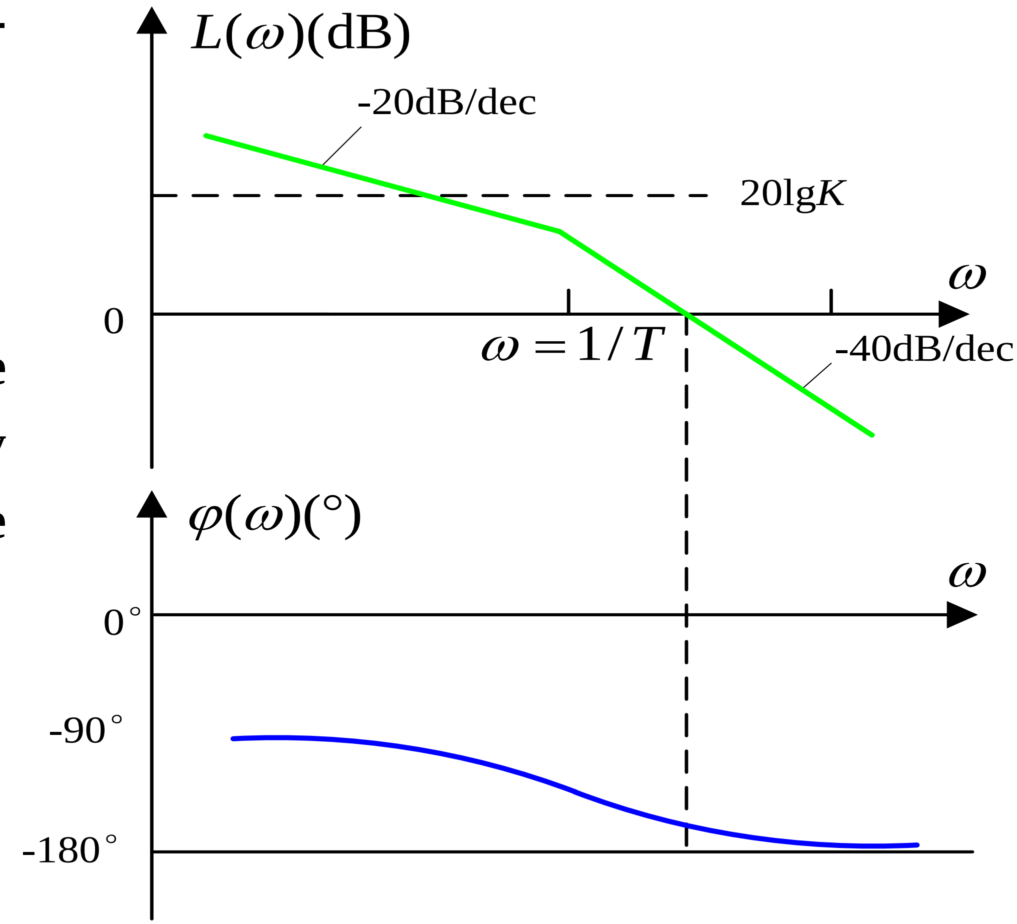
2. Stability criterion

- **The sufficient and necessary condition for the stability of closed-loop system: for the range of ω satisfying $L(\omega)\geq 0$, the number of times the phase diagram crosses the －180° line is $N=-P$ (the difference of crossing number in a clockwise and that in the counterclockwise, $N=2(N_{+}-N_{-})$), where P is the number of open-loop poles on the right-hand half s plane.**
- **For minimum phase systems, the sufficient and necessary condition for stability is that the difference of clockwise crosses and counterclockwise cross is zero, or $\varphi(\omega)$ does not cross －180° line.**

- **Example 6.14: The open-loop transfer function is**

$$G(s)H(s)=\frac{K}{s(Ts+1)}$$

Please apply the Bode diagram based stability criterion to determine the closed-loop stability.



伯德图

6-5 Relative Stability

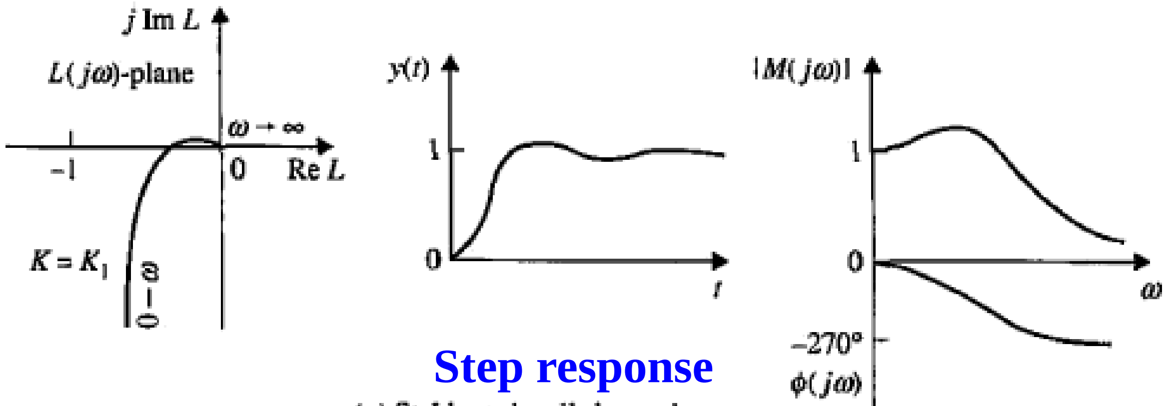
- **Absolute stability → relative stability (how stable)**
- **Time domain, relative stability is measured by parameters such as maximum overshoot and damping ratio.**
- **Frequency domain, the way of measuring relative stability is by how close the Nyquist diagram of $G(j\omega)H(j\omega)$ is to the $(-1,j0)$ point.**
- **Relative stability: Suppose that a system has no open-loop poles on the right-hand half s plane, and the closed-loop system is stable. The system is relatively more stable if the path of Nyquist diagram $G(j\omega)H(j\omega)$ is far away from $(-1,j0)$. If $G(j\omega)H(j\omega)$ crosses the point $(-1, j0)$, the closed-loop system is critically stable.**
- **Gain and phase margin: express the "closeness" of the path to the critical point.**

  Gain margin (GM)

  Phase margin (γ)

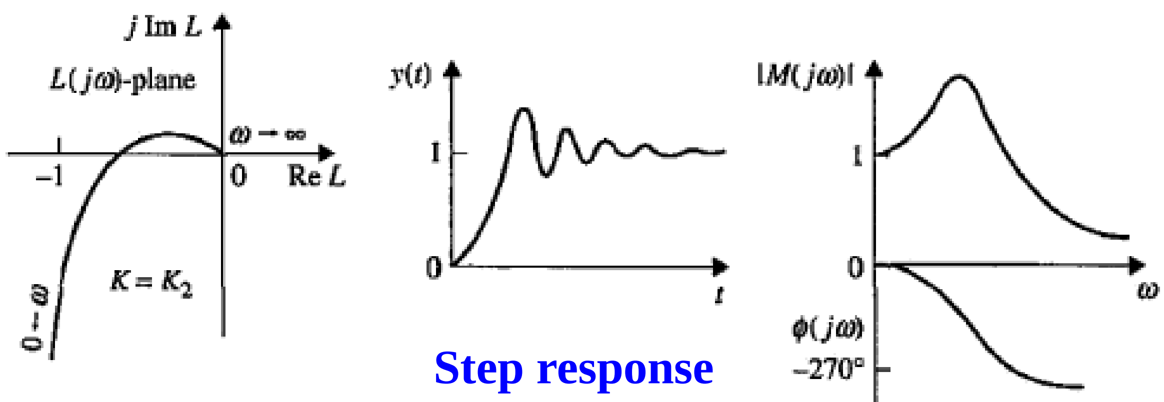
Relationship of system performance between time domain and frequency domain

stable



(a) Stable and well-damped system

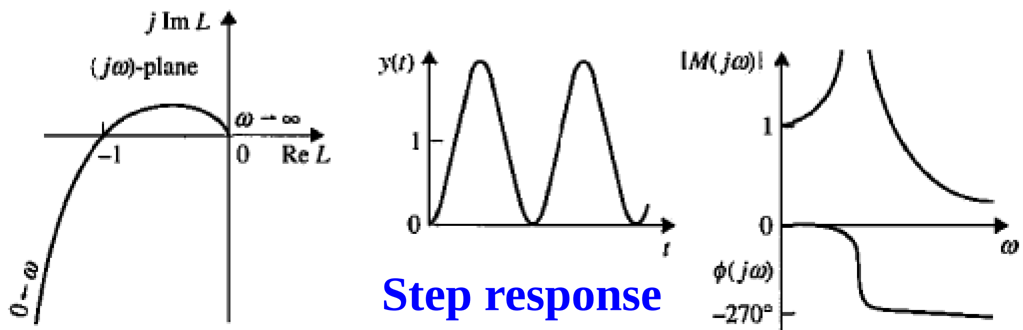
stable



(b) Stable but oscillatory system

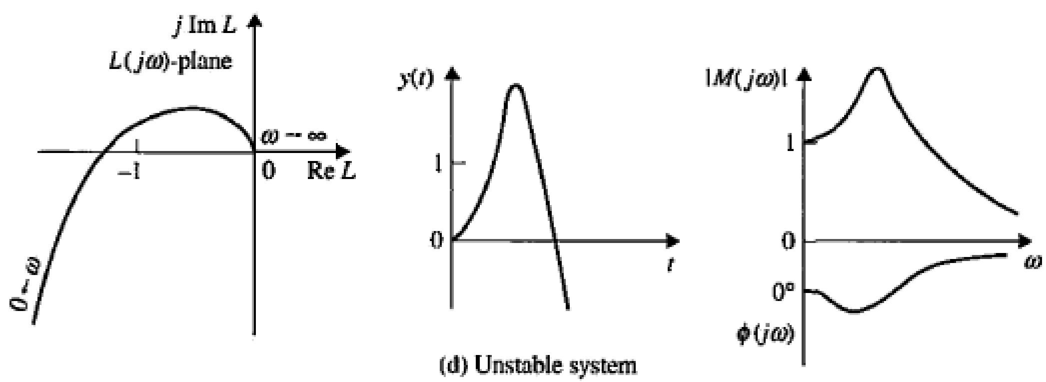
Relationship of system performance between time domain and frequency domain

Marginally stable



(c) Marginally unstable system

unstable



(d) Unstable system

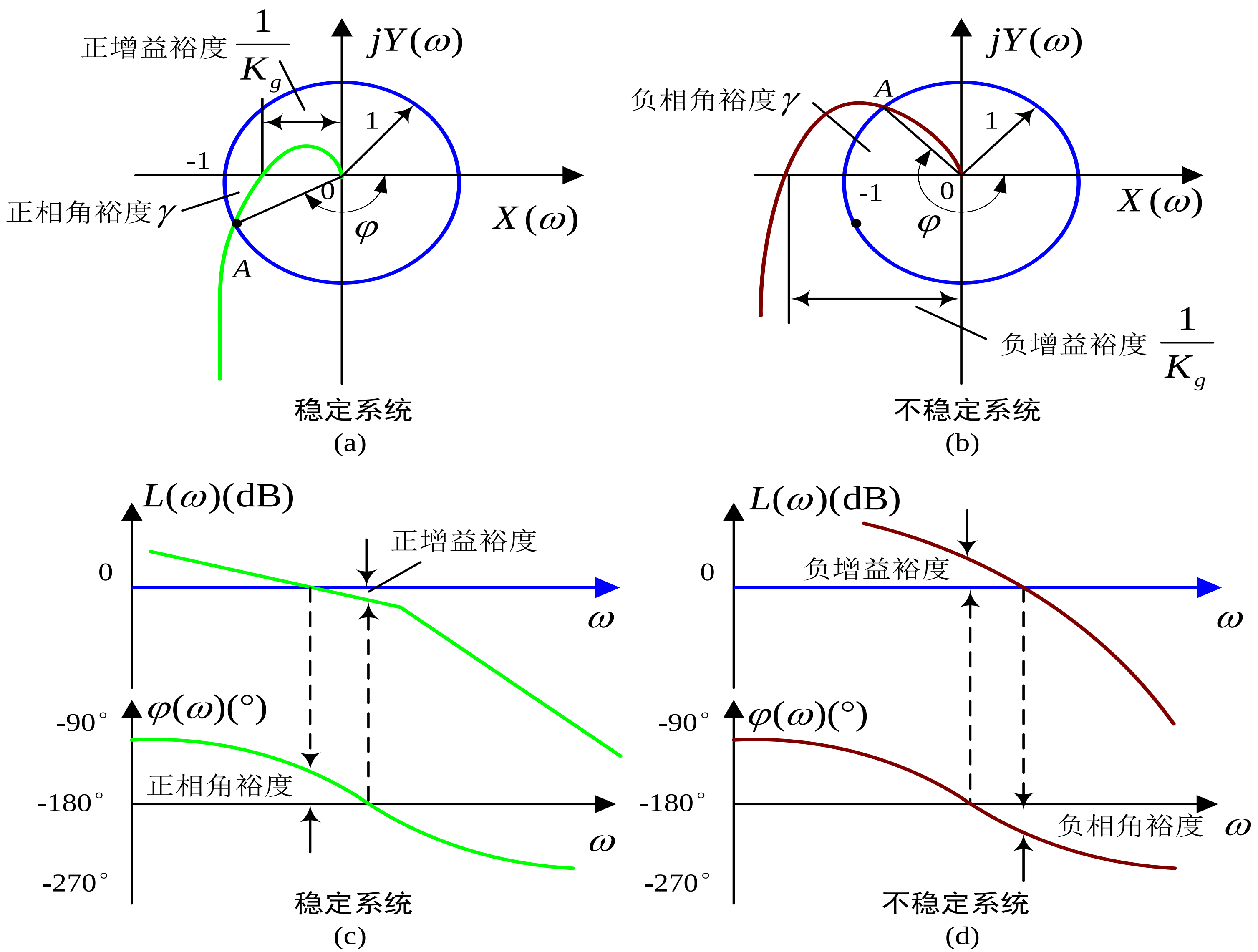
Step response

1. Phase margin γ

- **Definition: Phase margin is the amount of pure phase lag that has to be introduced to a system in order to make its frequency response pass through the critical point.**
- **The frequency ω_c satisfying $A(\omega) = 1$(i.e. $L(\omega)=0$) is gain crossover frequency (增益剪切角频率). The difference between the phase locus at the gain crossover frequency and the －180° line is phase margin.**

$$\gamma = \varphi(\omega_c) - (-180°) = 180° + \varphi(\omega_c)$$

- **If the phase margin is positive, the system is not only stable, but also allow γ phase lag to reach the critical stability.**



正增益裕度 $\frac{1}{K_g}$
$jY(\omega)$
1
-1
正相角裕度 γ
0
φ
$X(\omega)$
A
稳定系统
(a)
负相角裕度 γ
A
1
-1
0
φ
$X(\omega)$
负增益裕度 $\frac{1}{K_g}$
不稳定系统
(b)
$L(\omega)$(dB)
0
正增益裕度
ω
-90°
$\varphi(\omega)$(°)
正相角裕度
-180°
-270°
稳定系统
(c)
$L(\omega)$(dB)
0
负增益裕度
ω
-90°
$\varphi(\omega)$(°)
-180°
负相角裕度
-270°
不稳定系统
(d)

相角裕度和增益裕度

- **For a stable system，$\varphi(\omega_c)$ is above the －180° line. The phase margin is positive (see Fig. (c)).**
- **For an unstable system，$\varphi(\omega_c)$ is below the －180° line. The phase margin is negative (see Fig. (d))**
- **Correspondingly，γ is the phase difference from the crossover point A (of Nyquist diagram and the uinity circle) to the negative real axis. For stable systems, point *A* is below the negative real axis (see Fig. (a)). For unstable systems, point A is above negative real axis (see Fig. (b)).**

2. Gain Margin (*GM*)

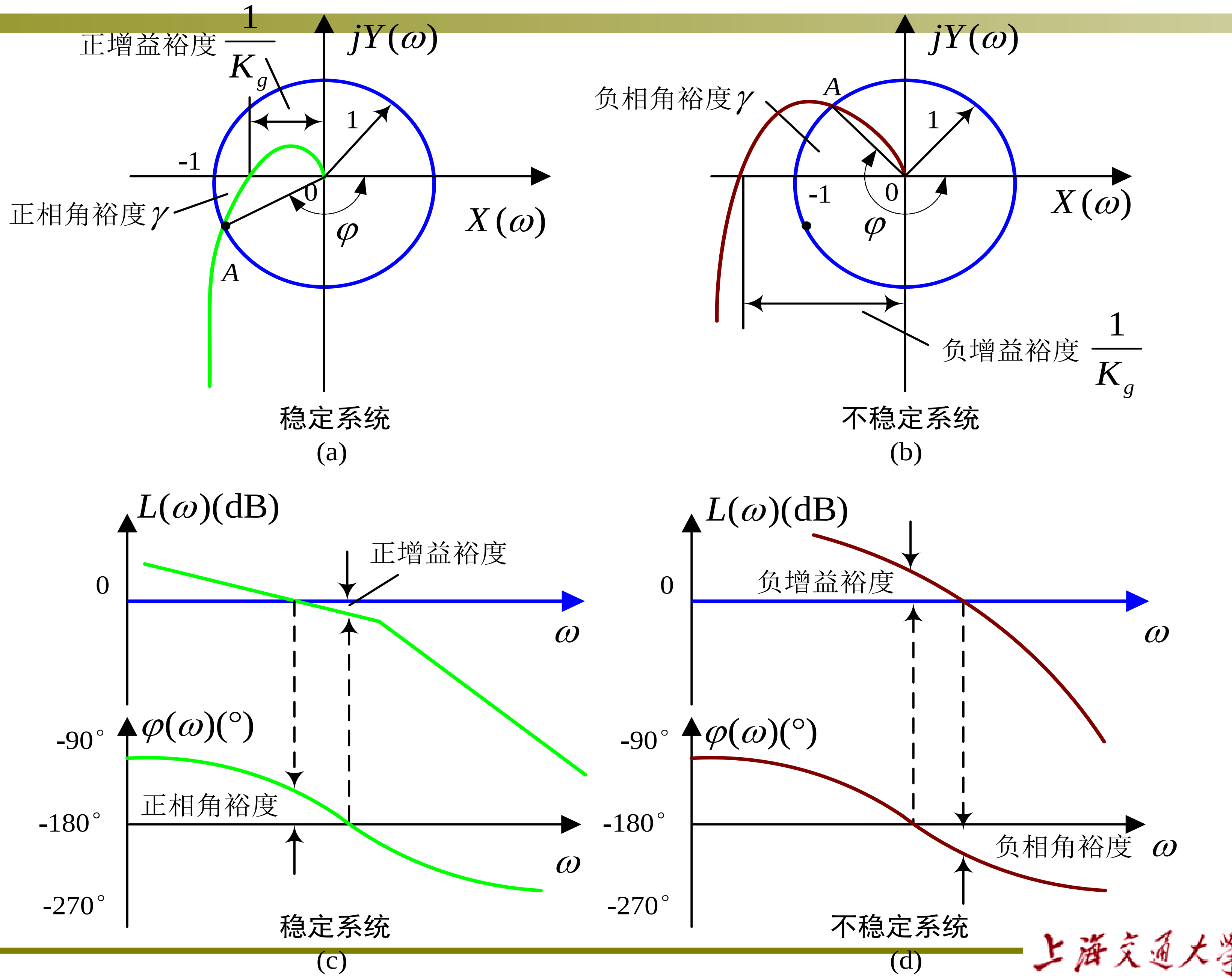
Phase crossover: A phase crossover on $G(j\omega)H(j\omega)$ is a point at which the diagram intersects the negative real axis.

Phase crossover frequency: ω_g is the frequency at the phase crossover. $\varphi(\omega_g)=-180°$

Definition of Gain Margin:

$$K_g = 1/A(\omega_g)$$ **Nyquist diagram**

$$GM = -20\lg A(\omega_g) = -L(\omega_g)$$ **Bode diagram**


正增益裕度 $\frac{1}{K_g}$
$jY(\omega)$
-1
1
0
正相角裕度 γ
φ
A
$X(\omega)$
稳定系统
(a)
负相角裕度 γ
A
$jY(\omega)$
1
-1
0
φ
$X(\omega)$
负增益裕度 $\frac{1}{K_g}$
不稳定系统
(b)
$L(\omega)$(dB)
0
正增益裕度
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-90°
正相角裕度
-180°
-270°
ω
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负增益裕度
ω
$\varphi(\omega)(°)$
-90°
-180°
负相角裕度
ω
-270°
不稳定系统
(d)

相角裕度和增益裕度

- **For a stable system, $L(\omega_g)$ in the Bode diagram is below *0*dB line. The gain margin is positive (see Fig. (c)). For an unstable system, $L(\omega_g)$ is above *0*dB line. The gain margin is negative (see Fig. (d)).**
- **Fig. (c) shows that the magnitude can be increased by *GM* dB, i.e. the open-loop gain can be increased by GM times to be marginally stable.**
- **In Nyquist diagram, the distance between the phase crossover point and the origin is $1/K_g$,*i.e.* $A(\omega_g)$.**

For stable systems, we have $K_g>1$ and GM>0dB (see Fig. (a)). For unstable systems, we have $K_g<1$ and GM<0dB (see Fig. (b)).

Remarks:

① For an stable minimum phase system, the phase margin is positive and gain margin is bigger than 0dB ;

② Being strictly, the relative stability should be given by both phase margin and gain margin.

But in engineering, the phase margin is more important.

For a good system with satisfactory stability margin, it is desired that

$$\gamma = 30^\circ \sim 60^\circ \qquad K_g \geq 2 \qquad GM \geq 6\text{dB}$$

④ **The phase margin requirement in engineering implies the slope at the gain crossover frequency should be bigger than －40dB/dec. In system design, we normally set it to be－20dB/dec。**

⑤ **Given a stable closed-loop system， the relative stability implies the time-domain performance. The bigger phase margin or gain margin, the less overshoot and oscillation and the more damping.**

- **Example 6.15: The open-loop transfer function of a unity feedback system is shown as follows:**

$$G(s)=\frac{K_1}{s(s+1)(s+5)}$$

Please obtain the phase margin and gain margin for K_1= 10 and K_1= 100.