Chapter 6: Frequency Domain Anaysis

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Outline

1. Introduction of Frequency Response
2. Frequency Response of the Typical Elements
3. **Nyquist and Bode Diagram of Open-loop System**
4. Nyquist-criterion
5. Frequency Response Based System Analysis
6. Frequency Response of Closed-loop Systems

Concept (1)

Graphical method (2–3)

Analysis (4–5)

6-3 Nyquist and Bode Diagrams for Open-loop System

- **Gain-phase characteristic diagram** — **Nyquist diagram**
- **Log-Magnitude characteristic diagram**

  ***L*-*ω* plot**

  Log-Phase characteristic diagram

  ***φ*-*ω* plot**

  } **Bode diagram**

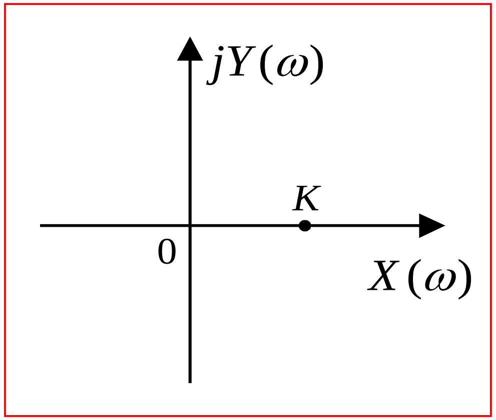
- **Partition of Open-loop System Into Typical Elements**
- **Sketching Method of Nyquist Diagram**
- **Sketching Method of Bode Diagram**
- **Minimum Phase System**

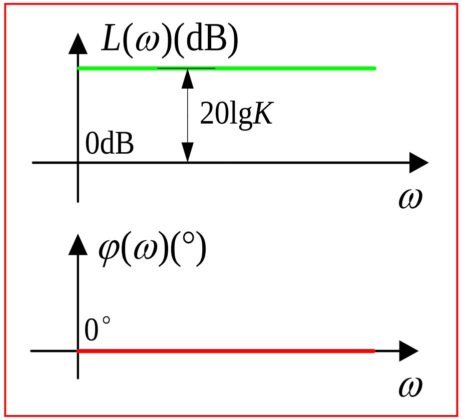
Review of Frequency Characteristics of Typical Elements

1.Proportional element

$$\begin{cases} A(\omega) = K \\ \varphi(\omega) = 0^\circ \end{cases}$$

$$\begin{cases} L(\omega) = 20\lg K \\ \varphi(\omega) = 0^\circ \end{cases}$$

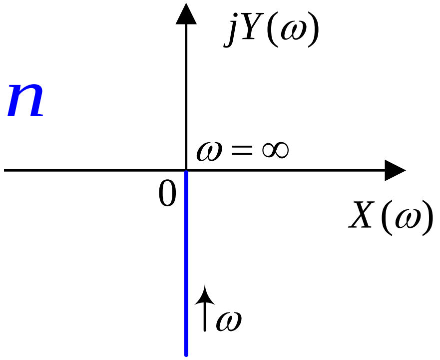


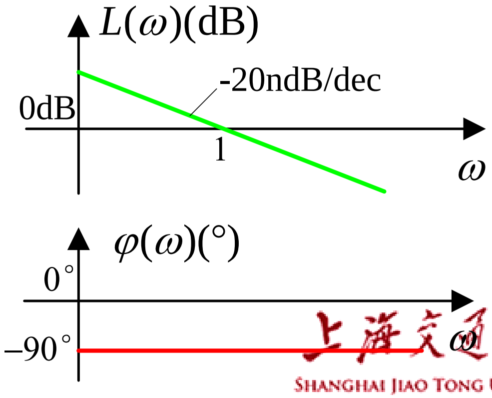


2. Integration element $\frac{1}{s^n}$

$$\begin{cases} A(\omega) = 1/\omega \\ \varphi(\omega) = -90^\circ \cdot n \end{cases}$$

$$\begin{cases} L(\omega) = -20\lg\omega \\ \varphi(\omega) = -90^\circ \cdot n \end{cases}$$





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3.Derivative Element s^n

$$\begin{cases} A(\omega)=\omega \\ \varphi(\omega)=90^\circ\cdot n \end{cases} \qquad \begin{cases} L(\omega)=20\lg\omega \\ \varphi(\omega)=90^\circ\cdot n \end{cases}$$

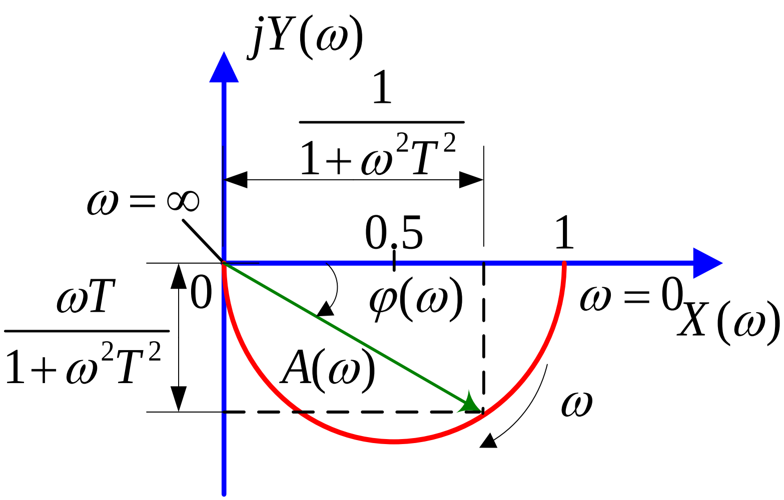
4.Inertial Element

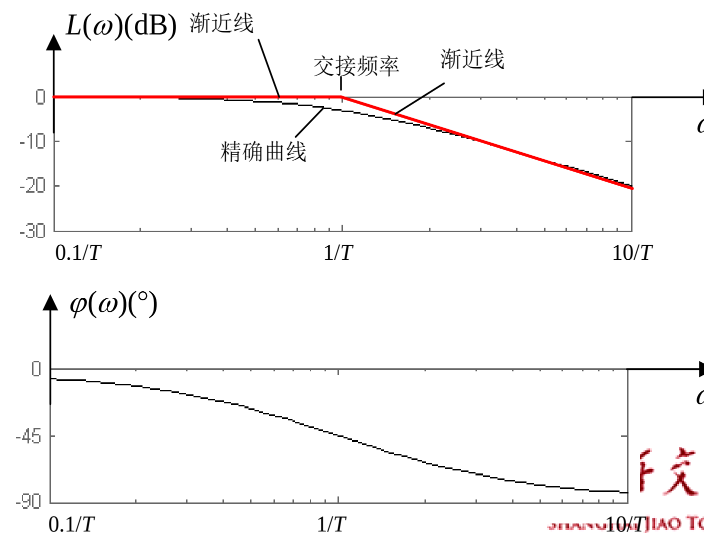
$$\begin{cases} A(\omega)=\dfrac{1}{\sqrt{1+\omega^2T^2}} \\ \varphi(\omega)=-\mathrm{arctg}\,\omega T \end{cases}$$

Asymptotes

$\omega \ll 1/T$, $L(\omega) \approx -20\lg 1=0$

$\omega \gg 1/T$, $L(\omega) \approx -20\lg\omega T$

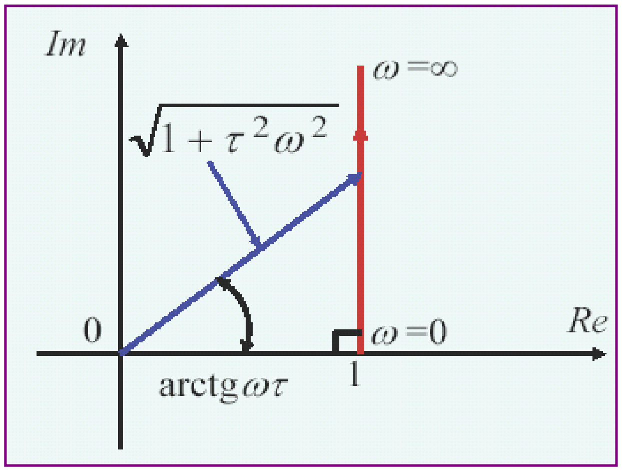


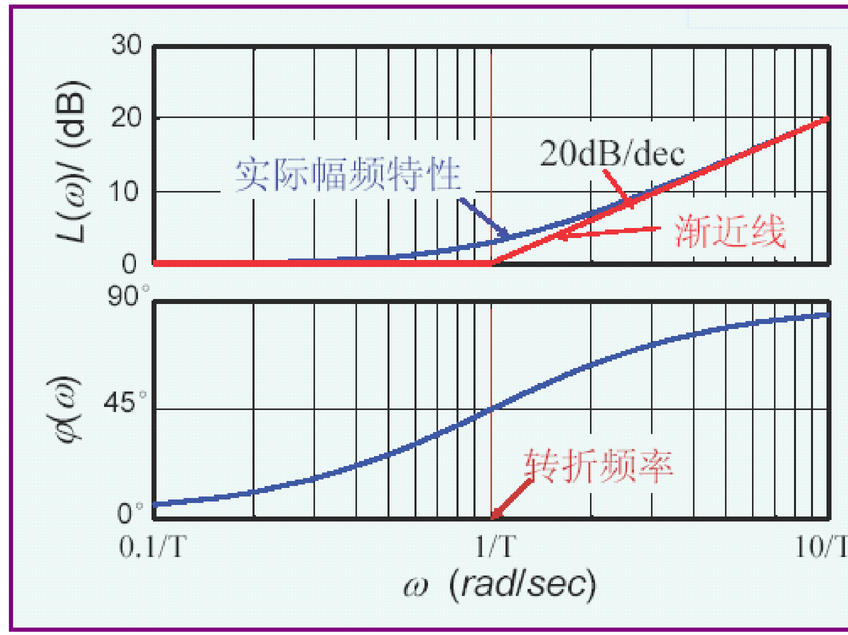


5. First-order Derivative Element

$$\begin{cases} A(\omega) = \sqrt{1+\omega^2 T^2} \\ \varphi(\omega) = \mathrm{arctg}\,\omega T \end{cases}$$

$$\begin{cases} L(\omega) = 20\lg\sqrt{1+\omega^2 T^2} \\ \varphi(\omega) = \mathrm{arctg}\,\omega T \end{cases}$$





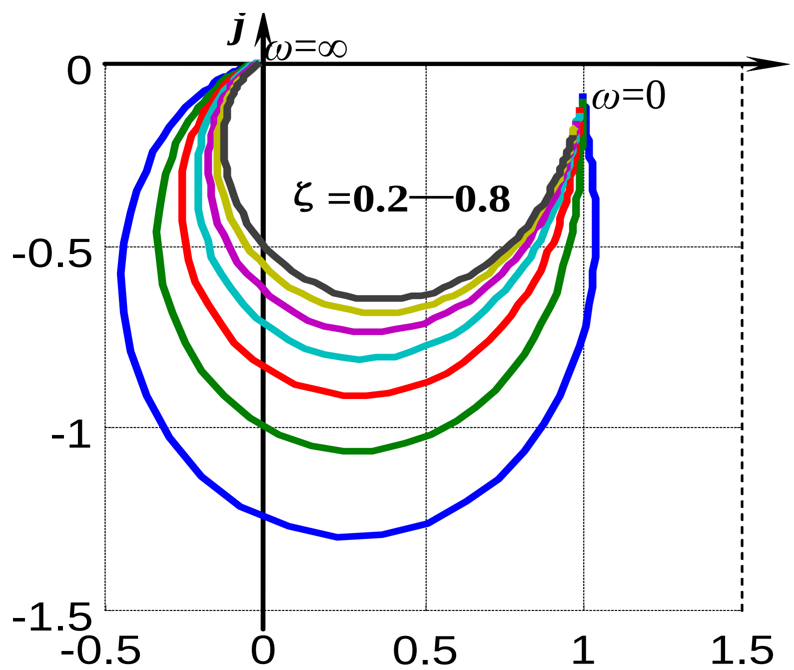
6. Second order oscillation element (Important)

$$G(j\omega) = \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n}}$$

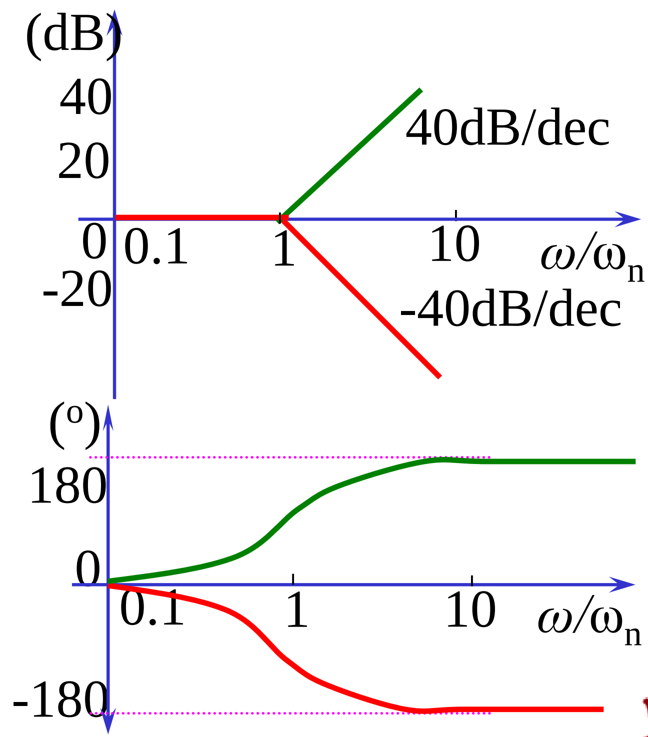
$$L(\omega) = -20\lg\sqrt{(1-\omega^2/\omega_n^2)^2 + 4\zeta^2(\omega/\omega_n)^2}$$

$$\varphi(\omega) = -arctg\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

- *For $\omega \ll \omega_n$, $L(\omega) \approx 0$*
- *For $\omega \gg \omega_n$, $L(\omega) \approx -40\lg\omega/\omega_n$*



Nyquist Diagram



Bode Diagram

6.3.1 Partition of Open-loop System Into Typical Elements

Assume that the open-loop transfer function contains several typical elements $G(s)=G_1(s)G_2(s)\cdots G_n(s)$

Frequency response of open-loop system is shown as follows:

$$G(j\omega)=G_1(j\omega)G_2(j\omega)\cdots G_n(j\omega)$$

$$A(\omega)e^{j\varphi(\omega)}=A_1(\omega)e^{j\varphi_1(\omega)}A_2(\omega)e^{j\varphi_2(\omega)}\cdots A_n(\omega)e^{j\varphi_n(\omega)}$$

Magnitude and Phase of open-loop system

$$\begin{cases} A(\omega)=A_1(\omega)\cdot A_2(\omega)\cdot\cdots\cdot A_n(\omega) \\ \varphi(\omega)=\varphi_1(\omega)+\varphi_2(\omega)+\cdots+\varphi_n(\omega) \end{cases}$$

Log-magnitude and phase are as follows:

$$\begin{cases} L(\omega)=L_1(\omega)+L_2(\omega)+\cdots+L_n(\omega) \\ \varphi(\omega)=\varphi_1(\omega)+\varphi_2(\omega)+\cdots+\varphi_n(\omega) \end{cases}$$

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6.3.2 Sketching Method of Nyquist Diagram

Requirement of sketching：

- **Don't need to be accurate**
- **Only need to determine the curve shape and key points (e.g. intercepts with axes)**
- **It is based on the partition of open-loop system into typical elements and relevant magnitude-phase characteristic diagrams.**

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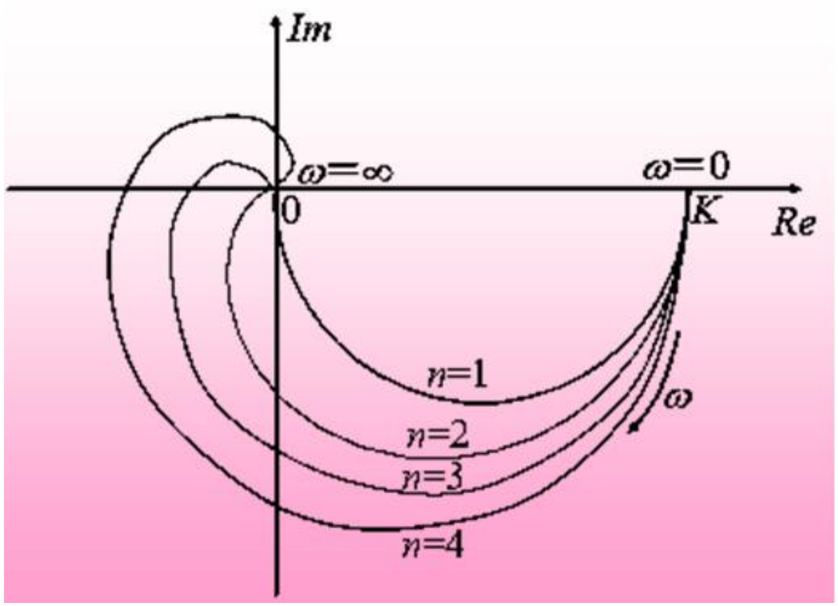
1. Nyquist Diagram of Typical Open-loop System

(1) Open-loop transfer function without integration and derivative elements

$$G(s)=\prod_{i=1}^{n}\frac{K}{T_i s+1} \qquad G(j\omega)=\prod_{i=1}^{n}\frac{K}{T_i j\omega+1}$$

n	$\omega=0$	$\omega=\infty$
1	$G(j0)=K\angle 0^\circ$	$G(j\infty)=0\angle -90^\circ$
2	$G(j0)=K\angle 0^\circ$	$G(j\infty)=0\angle -180^\circ$



Remark:

- **Contains *n* inertial elements and proportional element (*K>0*)**
- **Nyquist diagram starts from positive real axis and goes clockwise for *n* quadrant** $\omega=0\rightarrow\infty$

(2)Open-loop transfer function with first-order derivative element

$$G(s)=\frac{K\prod_{j=1}^{m}(\tau_j s+1)}{\prod_{i=1}^{n}(T_i s+1)}$$

For example: *m*=1, *n*=3

$$G(j\omega)=\frac{K(\tau_1 j\omega+1)}{(T_1 j\omega+1)(T_2 j\omega+1)(T_3 j\omega+1)}$$

$\omega=0 \quad G(j0)=K\angle 0^\circ$

$\omega=\infty \quad G(j\infty)=0\angle(90^\circ\ -3\times 90^\circ\ )=0\angle(-2\times 90^\circ\ )$

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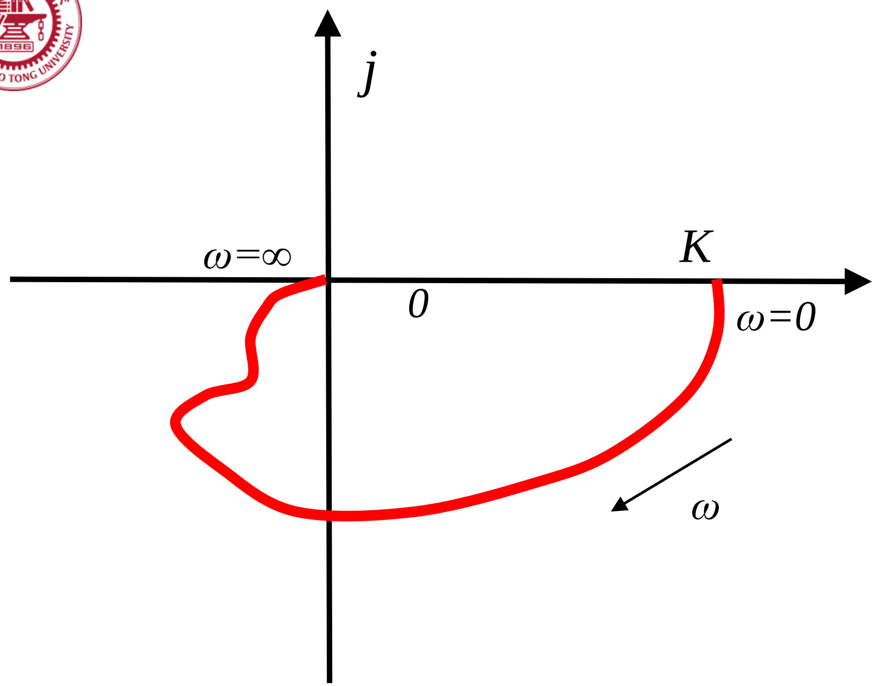

$m=1, n=3$, $T_1, T_2 > \tau_1 > T_3$

Remark: Given the open-loop transfer function with first-order derivative element, Nyquist diagram starts from positive real axis and appears concave and convex.

Diagram pattern

- **Given the open-loop transfer function with *m* first-order derivative elements and *n* inertial elements, Nyquist diagram has the following pattern**

$\omega=0$ $\quad G(j0)=K\angle 0°$

$\omega=\infty$ $\quad G(j\infty)=0\angle(m-n)\,90°$

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(3) Open-loop transfer function with integration elements

$$G(j\omega)=\frac{K(1+j\omega\tau_1)(1+j\omega\tau_2)\ldots(1+j\omega\tau_m)}{(j\omega)^{\upsilon}(1+j\omega T_1)(1+j\omega T_2)\ldots(1+j\omega T_{n-\upsilon})} \qquad n>m$$

Type I system（*v*＝1）

$\omega=0$	$A(0)=\infty$	$\varphi(0)=-90^{\circ}$
$\omega=\infty$	$A(\infty)=0$	$\varphi(\infty)=-(n-m)\times 90^{\circ}$

Type II system (*v* = 2)

$\omega=0$	$A(0)=\infty$	$\varphi(0)=-180^{\circ}$
$\omega=\infty$	$A(\infty)=0$	$\varphi(\infty)=-(n-m)\times 90^{\circ}$

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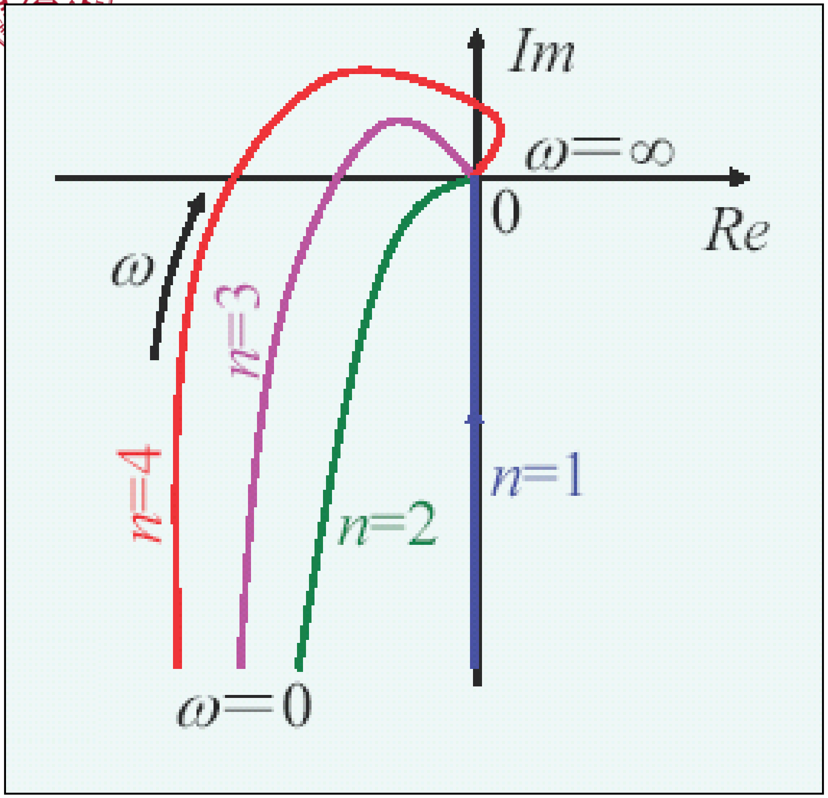
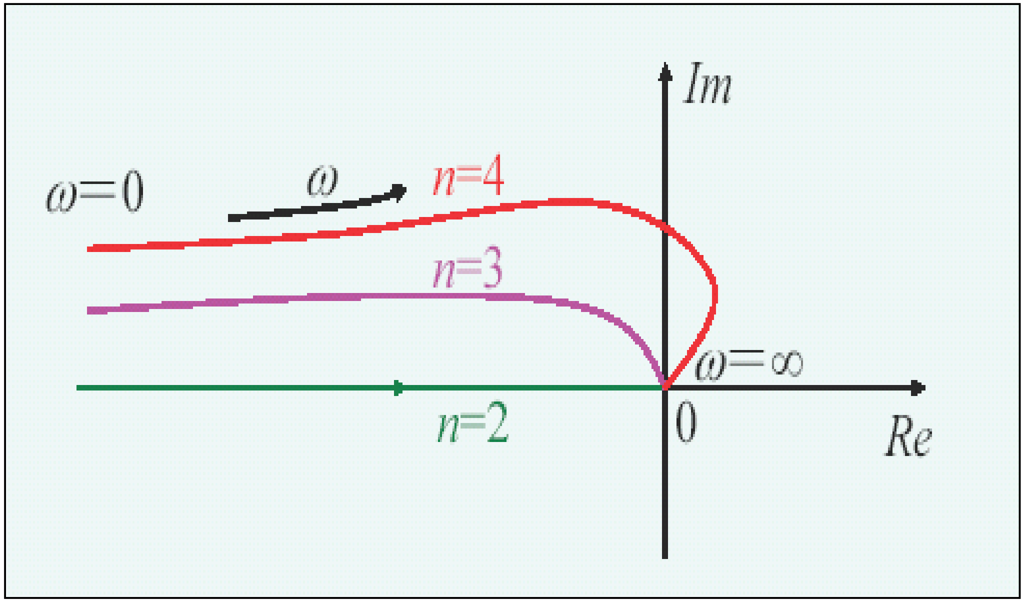


Type I open-loop system
with inertial elements only



Type II open-loop system
with inertial elements only

Remark: Given the open-loop transfer function with integration element, the Nyquist diagram starts from infinity.

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2. Rules for Open-loop Magnitude-Phase Characteristics

① For $\omega=0$，open-loop Nyquist diagram is determined by proportional and integration elements.

$v=0$	$G(j\omega)$ starts from positive real axis	$G(j0)=K\angle 0^\circ$
$v=1$	$G(j\omega)$ starts from the direction of negative imaginary axis	$G(j0)=\infty\angle -90^\circ$
$v=2$	$G(j\omega)$ starts from the direction of negative real axis	$G(j0)=\infty\angle -180^\circ$
$v=3$	$G(j\omega)$ starts from the direction of positive imaginary axis	$G(j0)=\infty\angle -270^\circ$

and so on

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② For $\omega=\infty$ and $n>m$，the magnitude of $G(j\omega)$ is 0，and phase is $-(n-m)\ 90°$, i.e.

$$G(j\infty)= 0\angle-(n-m)\ 90°$$

③ If the numerate of $G(s)$ include derivative elements，$G(j\omega)$ appears concave and convex with respect to ω. If there is no derivative element，$G(j\omega)$ is a smooth curve with respect to ω

④ Intercept with real axis is most important, and can be determined by the following method:

A. Solve $\mathrm{Im}[G(j\omega)]=0$ to get ω and then get $\mathrm{Re}[G(j\omega)]$;

B. Solve $\angle G(j\omega) = k\cdot 180°$ (k is an integer) to get Break Frequency. (转折频率，截止频率，穿越频率)

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3. Sketching of Nyquist Diagram

① Write down $A(\omega)$ and $\varphi(\omega)$;

② Get $G(j\omega)$ for $\omega = 0$ and $\omega = +\infty$ respectively

③ Get the intercept with real axis;

④ If necessary, determine the intercept with imaginary axis. It can be got by solving $\mathrm{Re}[G(j\omega)]=0$ or $\angle G(j\omega) = k\cdot 90^\circ$ (k is an integer) ;

⑤ If necessary, plot some points of the Nyquist diagram;

⑥ Sketch the curve.

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- **Example 6.3: Given the following open-loop transfer function, sketch the open-loop Nyquist diagram and determine the intercept of the diagram with real axis.**

$$G(s)H(s)=\frac{10}{s(0.5s+1)(0.2s+1)}$$

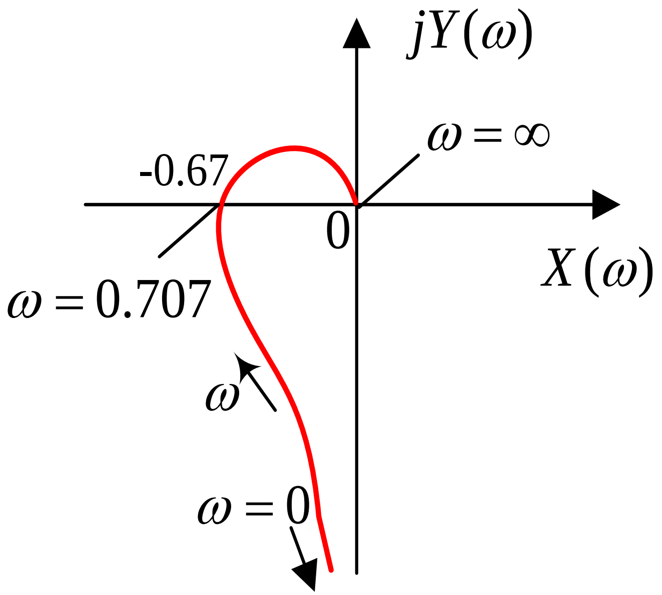
- **Example 6.4: Given the following open-loop transfer function, sketch the open-loop Nyquist diagram.**

$$G(s)H(s) = \frac{K(T_1 s + 1)}{s^2 (T_2 s + 1)}$$

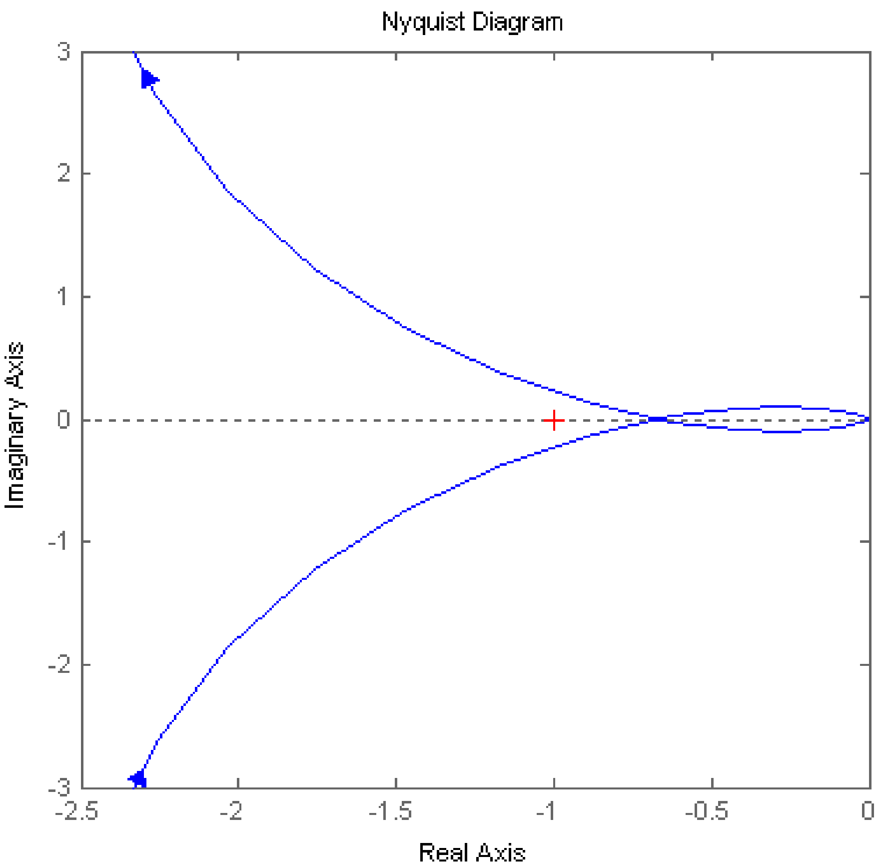
- **Example 6.5: Given the open-loop transfer function**

$$G(s)=\frac{1}{s(s+1)(2s+1)}$$

please sketch the open-loop Nyquist diagram.



Nyquist Diagram



Nyquist Diagram by using MATLAB

6.3.3 Open-loop Bode Diagram

- **1. Bode diagram**

$$G(s) = G_1(s)G_2(s)\cdots G_n(s)$$

$$\begin{cases} L(\omega) = L_1(\omega) + L_2(\omega) + \cdots + L_n(\omega) \\ \varphi(\omega) = \varphi_1(\omega) + \varphi_2(\omega) + \cdots + \varphi_n(\omega) \end{cases}$$

Bode diagram of an open-loop system is the superposition of the Bode diagrams of the typical elements.

The Log magnitude and phase of the open-loop system are the sum of those of the typical elements, respectively.

Note: Log magnitude characteristics of typical elements are shown by their asymptotes.

Steps for sketching *Bode Diagram*

1. Rewrite the frequency characteristics of an open-loop system into the production of typical elements.

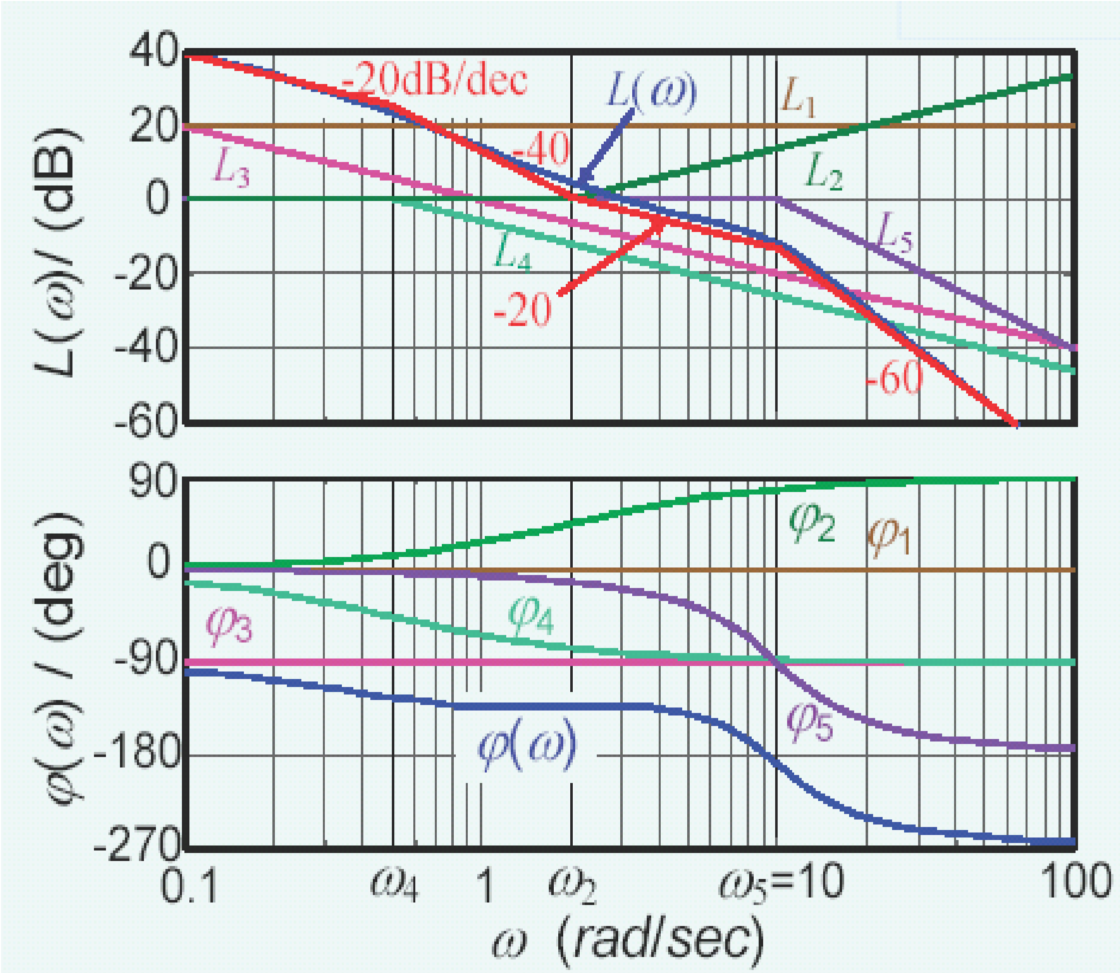
2. Determine the break frequencies and relevant slopes according to the typical elements.

Sketch rough Log magnitude asymptotes and phase curve.

3. If necessary, compensate the asymptotes to get more accurate Log-magnitude curves.

- **Example 6.6: Given the open-loop transfer function, please sketch the open-loop Bode Diagram.**

$$G(s)H(s) = \frac{1000(0.5s+1)}{s(2s+1)(s^2+10s+100)}$$



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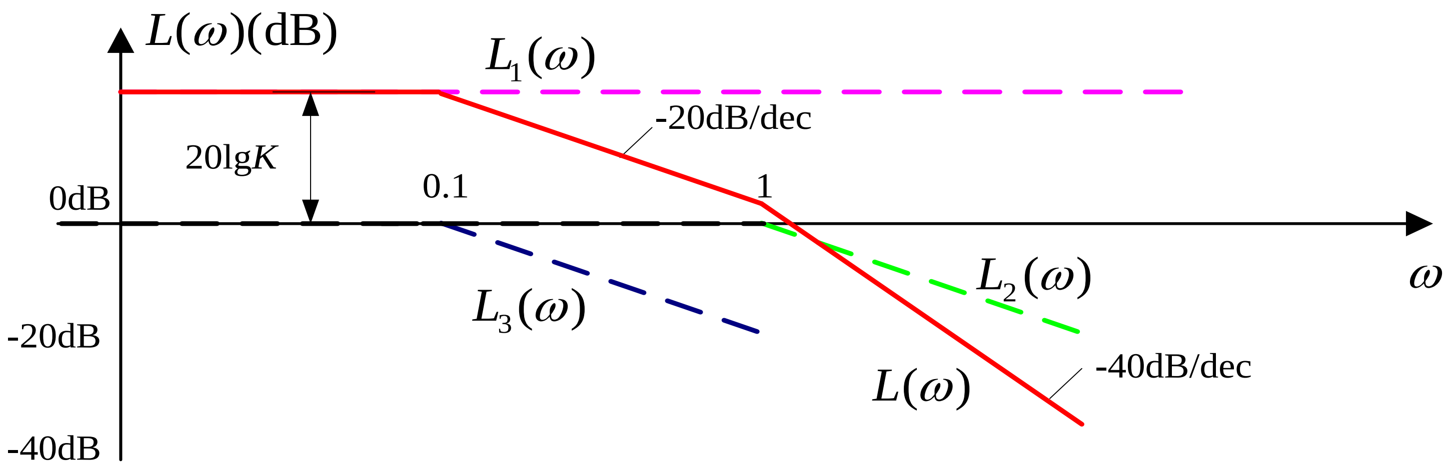
- **Example 6.7: Given the Type 0 open-loop system with the following transfer function, please sketch the Bode Diagram**

$$G(s)=\frac{K}{(1+s)(1+10s)}$$

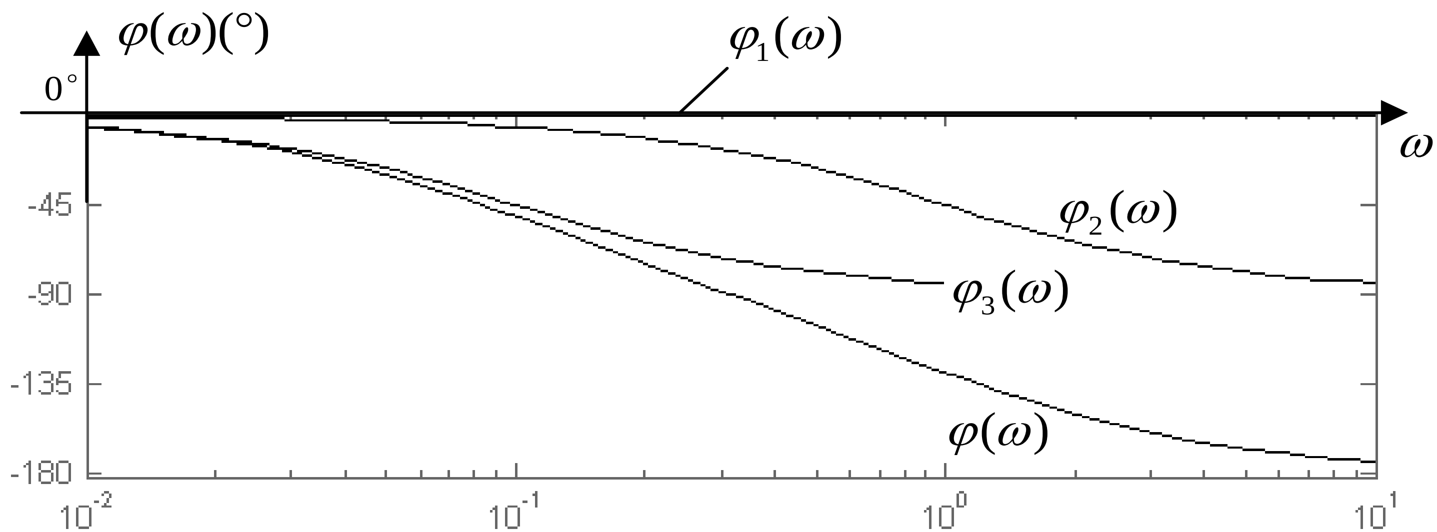
Solution: **The Log-magnitude and phase characteristics of the open-loop system are as follows:**

$$\begin{aligned}L(\omega)&=L_1(\omega)+L_2(\omega)+L_3(\omega)\\&=20\lg K-20\lg\sqrt{1+\omega^2}-20\lg\sqrt{1+100\omega^2}\end{aligned}$$

$$\begin{aligned}\varphi(\omega)&=\varphi_1(\omega)+\varphi_2(\omega)+\varphi_3(\omega)\\&=-\mathrm{arctg}\,\omega-\mathrm{arctg}10\omega\end{aligned}$$



$L(\omega)$(dB)
$L_1(\omega)$
-20dB/dec
20lg$K$
0dB
0.1
1
$L_2(\omega)$
ω
$L_3(\omega)$
-20dB
-40dB/dec
$L(\omega)$
-40dB



$\varphi(\omega)$(°)
$\varphi_1(\omega)$
0°
ω
-45
$\varphi_2(\omega)$
-90
$\varphi_3(\omega)$
-135
$\varphi(\omega)$
-180
10^{-2}
10^{-1}
10^{0}
10^{1}

2. Summary of Bode Diagram

① The slope of lower frequency line is $-20v$dB/dec, where v is the type of open-loop system. For $\omega=1$, $L(1)=20lgK$

② If there exist any break frequency less than 1, the point with $\omega=1$ and $L(1)=20lgK$ is on the extending line of lower frequency line.

③ The slops of Log-magnitude lines change with the break frequencies.

- For the element of $G(s)=(1+Ts)^{\pm 1}$, the change of slop at the break frequency is $\pm$20dB/dec;
- For second-order oscillation element, the change of slop at the break frequency is $-$40dB/dec .

④ Log-phase characteristics can be given by the sum of different typical elements, and also by computing $\varphi(\omega)$ directly.

Exercises

$$G(s) = \frac{0.1s + 1}{s(0.2s + 1)}$$