Chapter 6: Frequency Domain Anaysis

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Outline

1. Introduction of Frequency Response

Concept

2. Frequency Response of the Typical Elements
3. Bode Diagram of Open-loop System
4. Nyquist-criterion

Graphical method

5. Frequency Response Based System Analysis
6. Frequency Response of Closed-loop Systems

Analysis

6-1 Introduction

Key problem of control: stability and system performance

Question: Why Frequency Response

Analysis:

1. Weakness of root locus method relies on the existence of open-loop transfer function
2. Weakness of time-domain analysis method is that time response is very difficult to obtain
    - ☐ Computational complex
    - ☐ Difficult for higher order system
    - ☐ Difficult to partition into main parts
    - ☐ Not easy to show the effects by graphical method

Frequency Response Analysis

Three advantages:

* Frequency response(mathematical modeling) can be obtained directly by experimental approaches.

* Easy to analyze effects of the system with sinusoidal signals

* Convenient to measure system sensitivity to noise and parameter variations

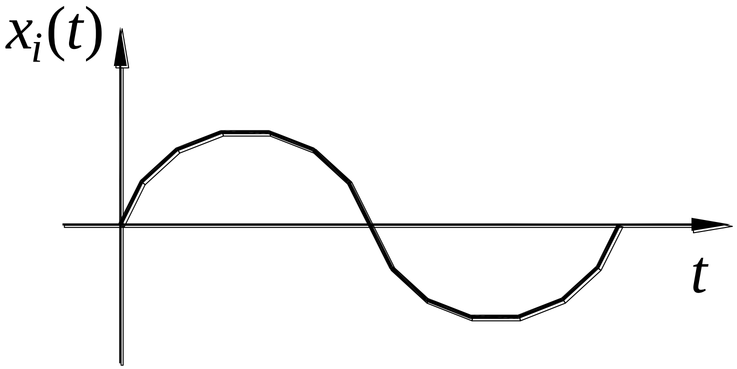
However, NEVER be limited to sinusoidal input

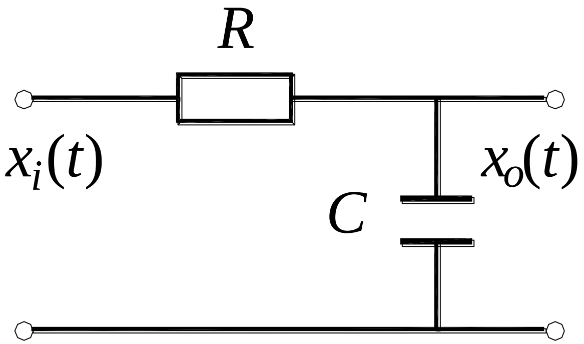
* Frequency-domain performances → time-domain performances

Frequency domain analysis is a kind of (indirect method) engineering method. It studies the system based on frequency response which is also a kind of mathematical model.

Frequency Response

Example 5.1: RC circuit





$$u_i = A\sin(\omega t + \varphi_0)$$

$$U_i(s) = \frac{U_{1m}\omega}{s^2 + \omega^2}$$

$$U_0(s) = \frac{1}{Ts+1} \cdot \frac{U_{1m}\omega}{s^2 + \omega^2}$$

Transfer function

$$G(s) = \frac{1}{RCs+1} = \frac{1}{Ts+1}$$

By using inverse Laplace transform

$$u_0(t) = \underbrace{\frac{U_{1m}T^2\omega}{1 + T^2\omega^2}}_{\textbf{Transient response}} e^{-\frac{t}{T}} + \underbrace{\frac{U_{1m}}{\sqrt{1 + T^2\omega^2}}\sin(\omega t + \varphi)}_{\textbf{Steady state response}} \quad \varphi = \text{-arctg}(\omega T)$$

Steady state response of u_o

$$\lim_{t \to \infty} u_0 = \frac{U_{1m}}{\sqrt{1+T^2\omega^2}}\sin(\omega t + \varphi) \qquad suppose\ \varphi_0 = 0$$

$$= U_{1m}\left|\frac{1}{1+j\omega T}\right|\sin(\omega t + \angle\frac{1}{1+j\omega T})$$

Proposition: When the input to a linear time-invariant (LTI) system is sinusoidal, the steady-state output is a sinusoid with the same frequency but possibly with different amplitude and phase.

Definition：Frequency response (or characteristic) is the ratio of the complex vector of the steady-state output versus sinusoidal input for a linear system.

$$\boxed{G(j\omega)=\frac{1}{1+j\omega T}=A(\omega)e^{j\varphi(\omega)}}$$

$$A(\omega)=\left|\frac{1}{1+j\omega T}\right|=\frac{1}{\sqrt{1+\omega^2T^2}};$$

Magnitude response

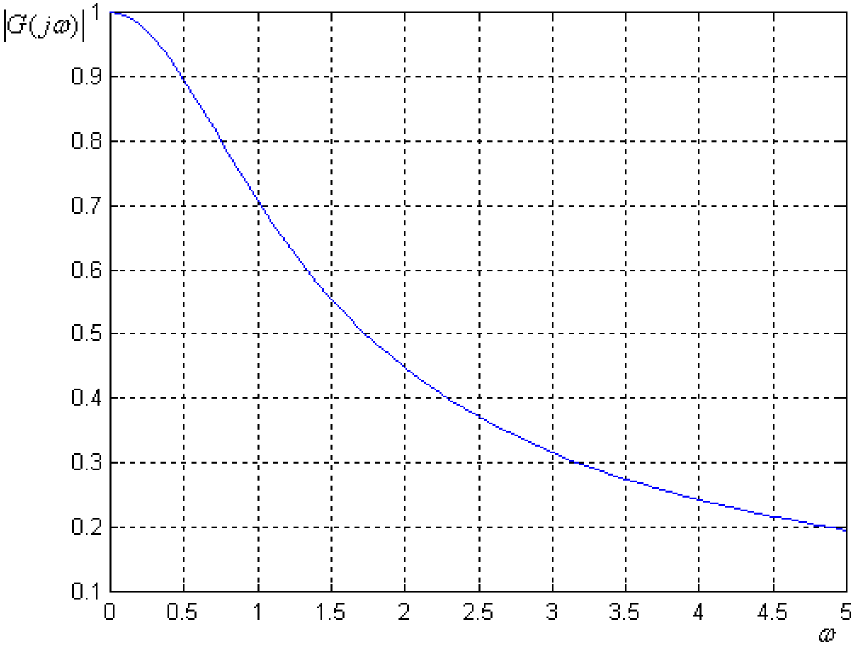
$$\phi(\omega)=\angle(\frac{1}{1+j\omega T})=-\mathrm{arctg}\,\omega T$$

Phase response

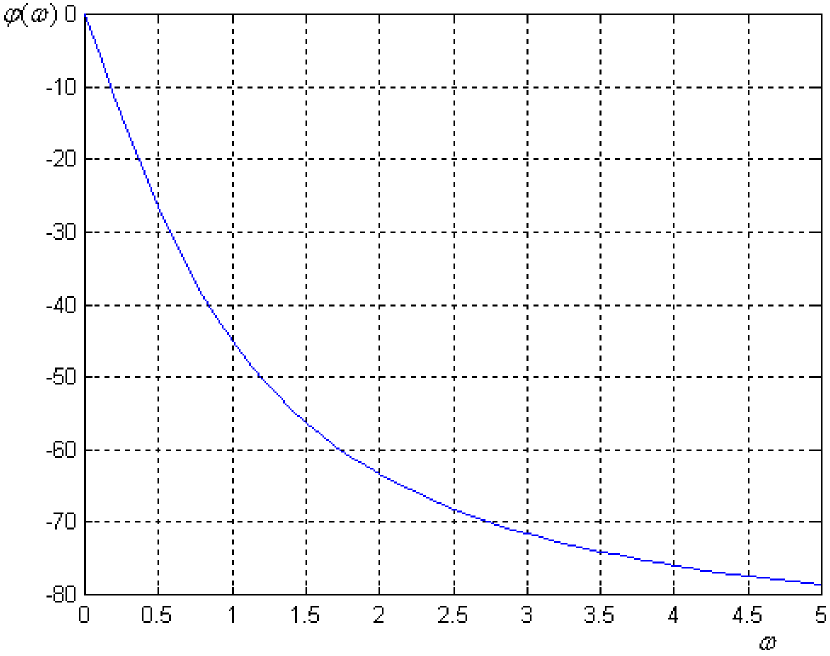
$\omega=0$ **The output has same magnitude and phase with input**

Magnitude will be attenuated and phase lag is increased.

$\omega\uparrow$



Magnitude of output versus input



Phase error of output and input

Magnitude characteristic

Phase characteristic

Generalized to linear time-invariant system

Transfer function of closed-loop system

$$G(s)=\frac{C(s)}{R(s)}=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where $p_1,\cdots,p_n$ are different closed-loop poles.

Given the sinusoidal input

$$r(t)=A_r\sin\omega t \qquad R(s)=\frac{A_r\omega}{s^2+\omega^2}$$

$$C(s)=G(s)\cdot\frac{A_r\omega}{s^2+\omega^2}=\frac{N(s)}{D(s)}\cdot\frac{A_r\omega}{s^2+\omega^2}$$

$$=\frac{a}{s+j\omega}+\frac{\bar{a}}{s-j\omega}+\frac{b_1}{s+p_1}+\frac{b_2}{s+p_2}+\ldots+\frac{b_n}{s+p_n}$$

$$c(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1 e^{-p_1 t} + b_2 e^{-p_2 t} + \ldots + b_n e^{-p_n t}$$

$$= \boxed{\sum_{i=1}^{n} b_i e^{-p_i t}} + \boxed{(ae^{-j\omega t} + \bar{a}e^{j\omega t})}$$

$$= c_t(t) + c_s(t) \qquad (t \geq 0)$$

Transient response **Steady state response**

For a stable closed-loop system, we have $-p_i < 0$

$$a = G(s) \cdot \frac{A_r \omega}{s^2 + \omega^2} \cdot (s + j\omega)\big|_{s=-j\omega} = -\frac{A_r G(-j\omega)}{2j}$$

$$\bar{a} = G(s) \cdot \frac{A_r \omega}{s^2 + \omega^2} \cdot (s - j\omega)\big|_{s=j\omega} = \frac{A_r G(j\omega)}{2j}$$

$$G(j\omega) = |G(j\omega)|\, e^{j\angle G(j\omega)} \qquad G(-j\omega) = |G(-j\omega)|\, e^{-j\angle G(j\omega)} = |G(j\omega)|\, e^{-j\angle G(j\omega)}$$

Furthermore, we have

$$
\begin{aligned}
c_s(t) &= ae^{-j\omega t} + \bar{a}e^{j\omega t} \\
&= A_r\,|G(j\omega)|\,\frac{e^{j(\omega t+\angle G(j\omega))} - e^{-j(\omega t+\angle G(j\omega))}}{2j} \\
&= A_r\,|G(j\omega)|\sin(\omega t+\angle G(j\omega)) \\
&= A_c\sin(\omega t+\varphi)
\end{aligned}
$$

The magnitude and phase of steady state are as follows

$$A_c = A_r\,|G(j\omega)|; \qquad \varphi = \angle G(j\omega)$$

By knowing the transfer function G(s) of a linear system, the magnitude and phase characteristics completely describe the steady-state performance when the input is sinusoid.

Frequency-domain analysis can be used to predict both time-domain transient and steady-state system performance.

Relation of transfer function and frequency characteristic of LTI system (only for LTI system)

$$\boxed{G(s)\big|_{s=j\omega} = G(j\omega) = |G(j\omega)|\, e^{j\angle G(j\omega)}}$$

- Substitute *s=jω* into the transfer function

$$F(s) = L[f(t)] = \int_{-\infty}^{+\infty} f(t)e^{-st}dt$$

Laplace transform

$$f(t) = L^{-1}[F(s)] = \int_{c-\infty}^{c+\infty} F(s)e^{st}ds$$

Inverse Laplace transform

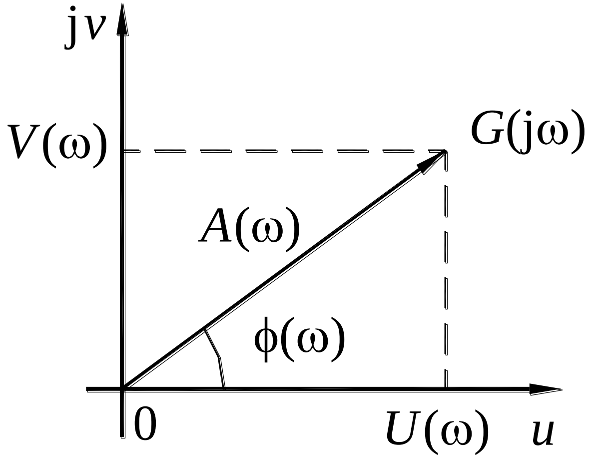
$$F(j\omega) = F\left[f(t)\right] = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

Fourier transform

$$f(t) = F^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}d\omega$$

Inverse Fourier transform

Vector of frequency characteristics



It is a complex vector, and has three forms:

Algebraic form

$$G(j\omega)=U(\omega)+jV(\omega)$$

Polar form

$$G(j\omega)=\left|G(j\omega)\right|\angle G(j\omega)=A(\omega)\angle\phi(\omega)$$

Exponential form $G(j\omega)=\left|G(j\omega)\right|e^{j\angle G(j\omega)}=A(\omega)e^{j\phi(\omega)}$

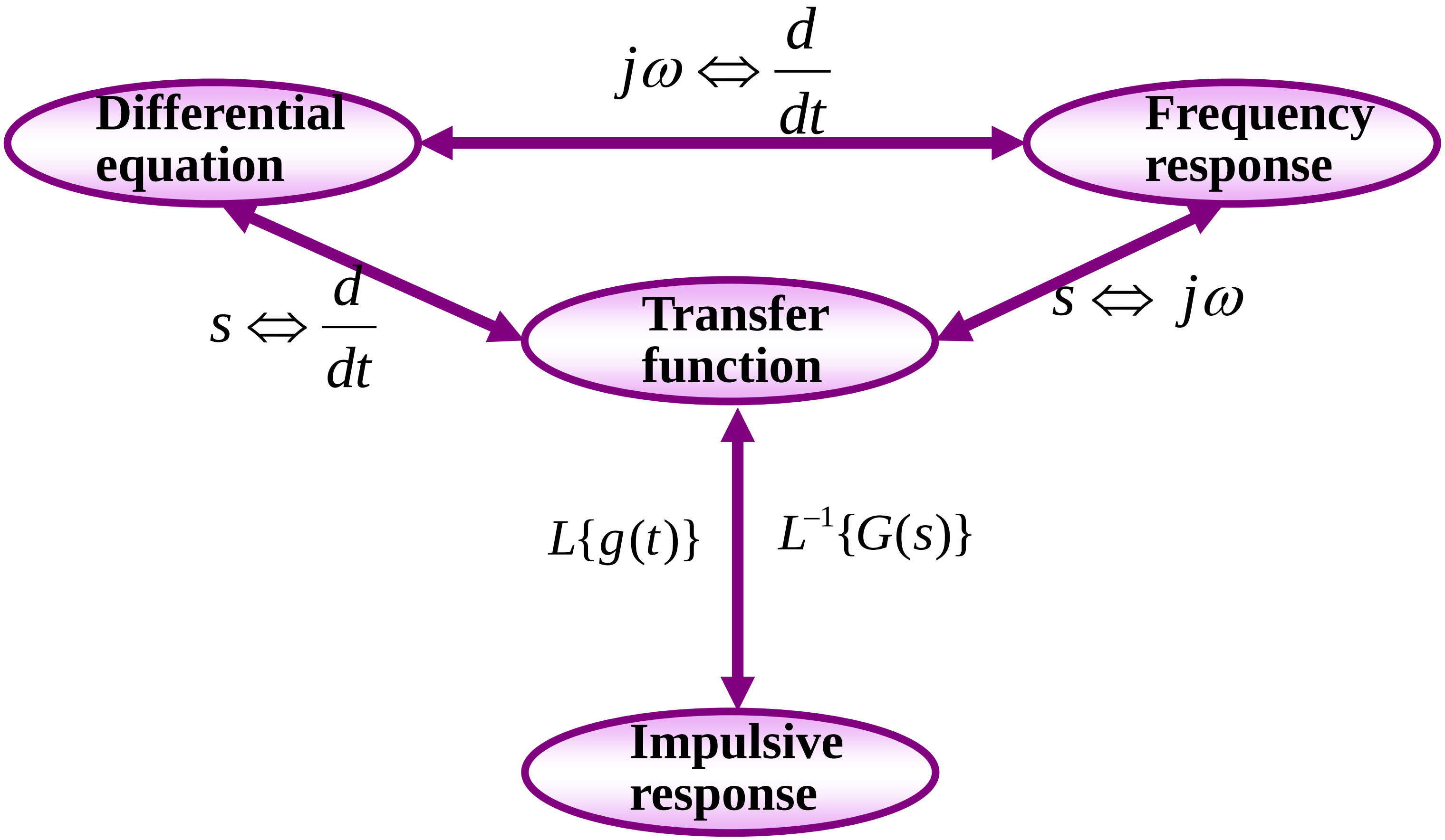
$$A(\omega)=\left|G(j\omega)\right|=\sqrt{U^2(\omega)+V^2(\omega)}$$

$$U(\omega)=A(\omega)\cos\phi(\omega)$$

$$\phi(\omega)=\arctan\left[\frac{V(\omega)}{U(\omega)}\right]$$

$$V(\omega)=A(\omega)\sin\phi(\omega)$$

We have learned following mathematical models：
differential equation, transfer function and frequency response



$$j\omega \Leftrightarrow \frac{d}{dt}$$

Differential equation

Frequency response

$$s \Leftrightarrow \frac{d}{dt}$$

Transfer function

$$s \Leftrightarrow j\omega$$

$$L\{g(t)\}$$

$$L^{-1}\{G(s)\}$$

Impulsive response

Example 5.2: Given the transfer function

$$G(s)=\frac{C(s)}{R(s)}=\frac{1}{s^{2}+3s+4}$$

Differential equation:

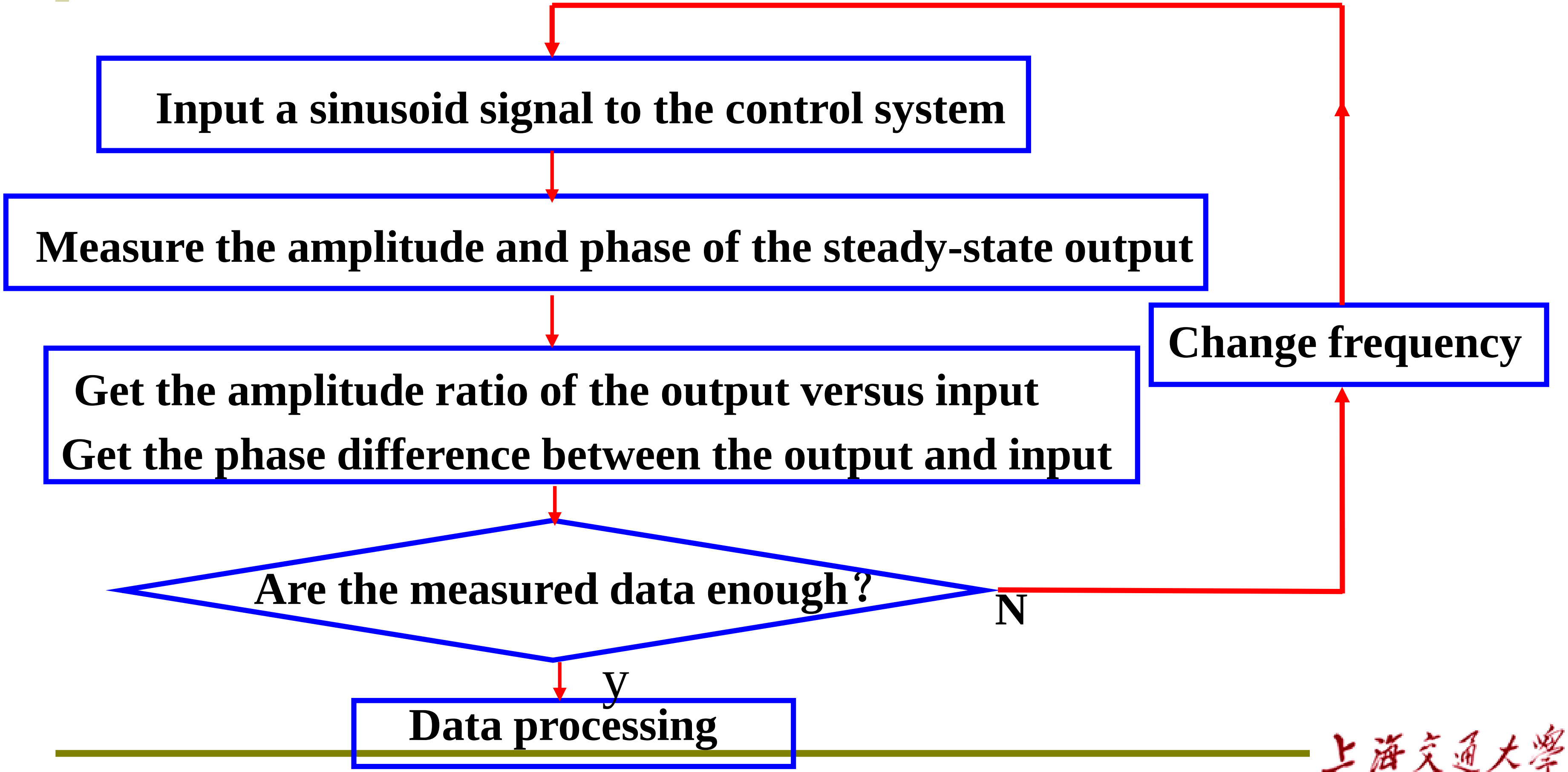
$$\frac{d^{2}c(t)}{dt^{2}}+3\frac{dc(t)}{dt}+4c(t)=r(t)$$

Frequency response:

$$G(j\omega)=\frac{c(j\omega)}{s(j\omega)}=\frac{1}{(j\omega)^{2}+3(j\omega)+4}=\frac{1}{4-\omega^{2}+3j\omega}$$

6-2 Frequency Characteristics of Typical Elements of system

How to get frequency characteristic?



Input a sinusoid signal to the control system

Measure the amplitude and phase of the steady-state output

Get the amplitude ratio of the output versus input

Get the phase difference between the output and input

Are the measured data enough？

N

Change frequency

y

Data processing

Nyquist Diagram

- **Magnitude characteristic diagram**
  ***A*-*ω* plot**
- **Phase characteristic diagram**
  ***φ*-*ω* plot**

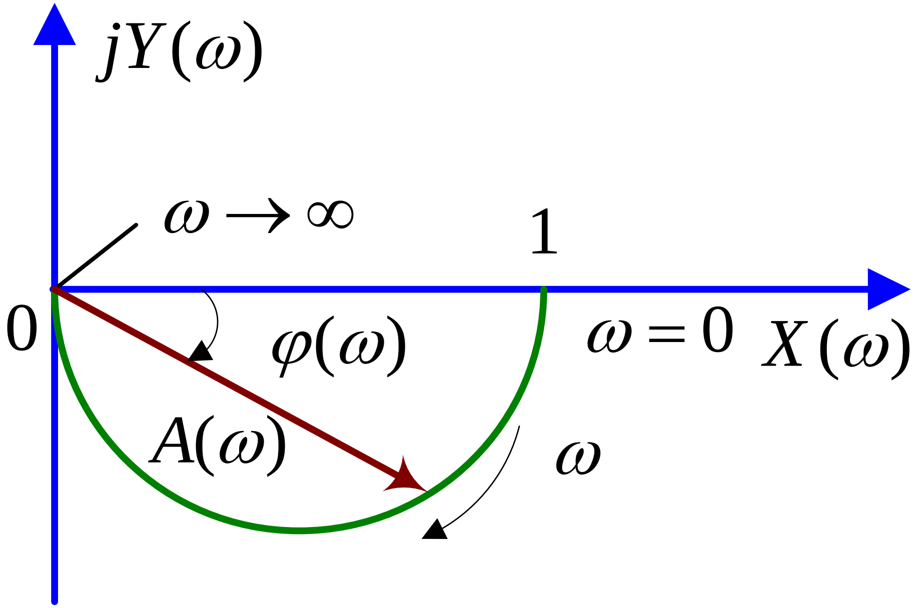
Bode diagram

- **Gain-phase characteristic diagram** —— **Nyquist diagram**

Polar form or algebraic form: *A* and *φ* define a vector for a particular frequency *ω*.

$$\boxed{G(j\omega)=\left|G(j\omega)\right|e^{j\varphi(\omega)}=X(\omega)+jY(\omega)}$$

ω	0	$1/(2\tau)$	$1/\tau$	$2/\tau$	$3/\tau$	$4/\tau$	$5/\tau$	∞
$A(\omega)$	1	0.89	0.707	0.45	0.32	0.24	0.2	0
$\varphi(\omega)$	0	-26.6	-45	-63.5	-71.5	-76	-78.7	-90



Nyquist Diagram of RC Circuit

Bode Diagram

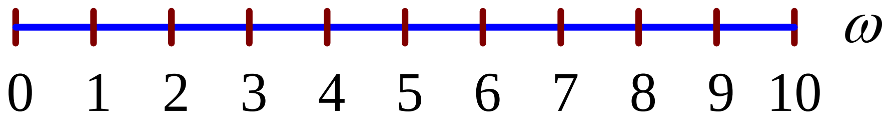
- Bode Diagram: Logarithmic plots of magnitude response and phase response
- Horizontal axis: *lgω* (logarithmic scale to the base of 10)（unit: rad/s）
- Log Magnitude

In feedback-system, the unit commonly used for the logarithm of the magnitude is the decibel (dB)

$$\boxed{L(\omega)=20\lg|G(j\omega)|=20\lg A(\omega)}$$

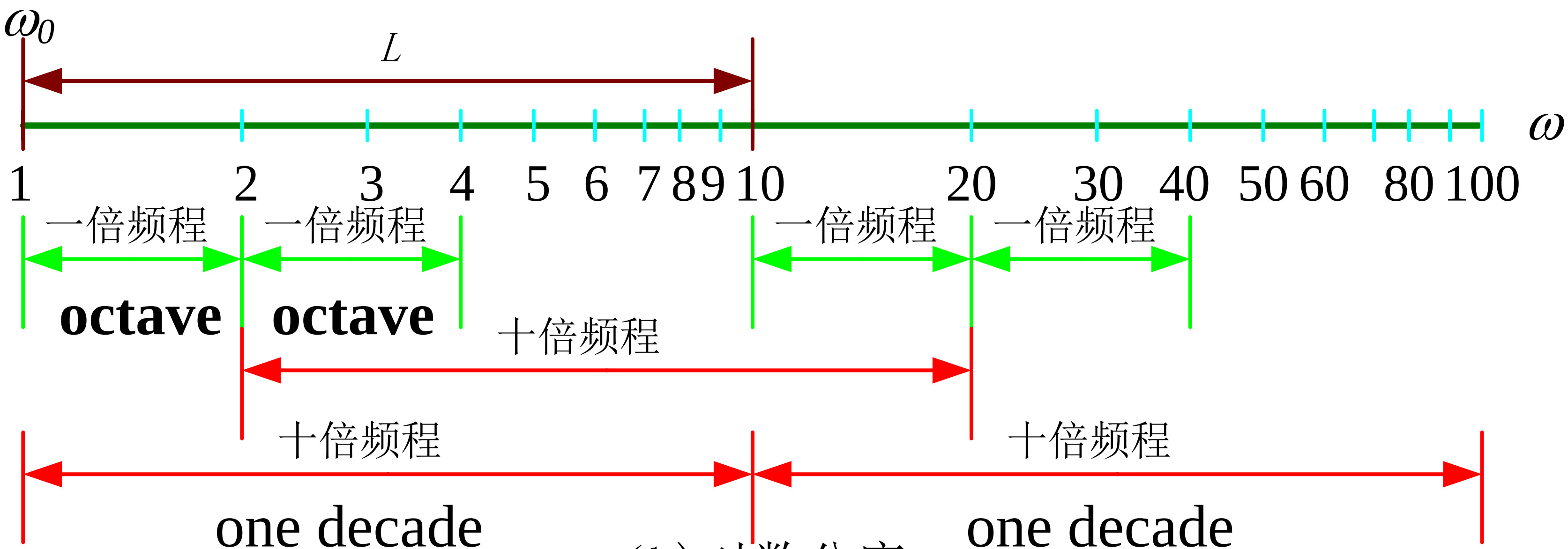
Property 1: As the magnitude doubles, the decibel value increases by 6 dB.
As the magnitude increases by a factor of 10, the decibel value increases by 20 dB.

- **Log scale and linear scale**



one decade

(a)线性分度

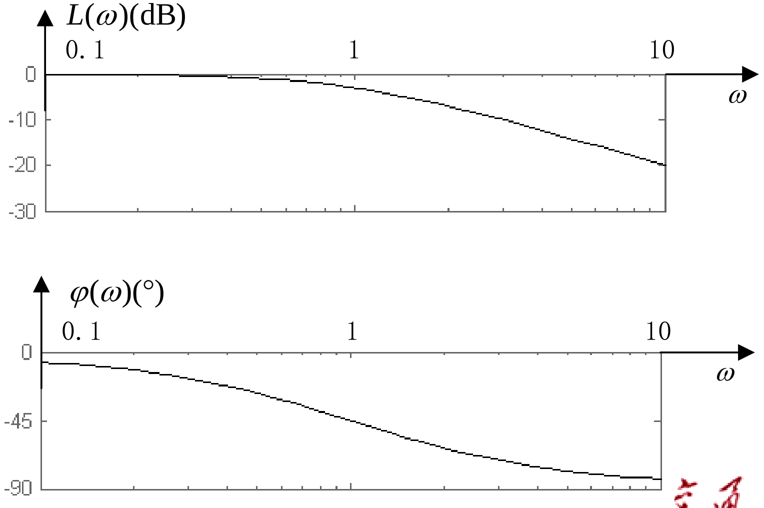


(b)对数分度

- **Notation:**
  - **Logarithmic scale use the nonlinear compression of horizontal scale. It can reflect a large region of frequency variation. Especially expand the low-frequency range.**
  - **Logarithmic magnitude response simplify the plotting. Multiplication and division are changed into addition and subtraction.**
  - **We cannot sketch $\omega = 0$ on the horizontal scale. The smallest ω can be determined by the region of interest.**

- **Given T=1， plot the Bode Diagram by using Matlab**

  bode([1],[1 1])

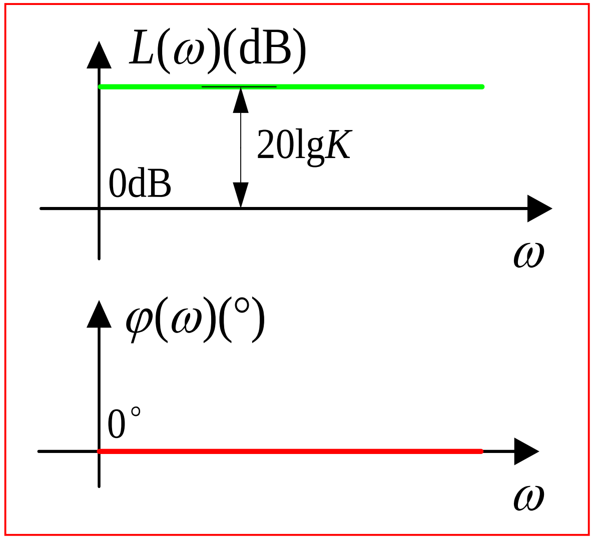


Frequency Characteristic of Typical Elements

- Seven typical elements

1.Proportional element

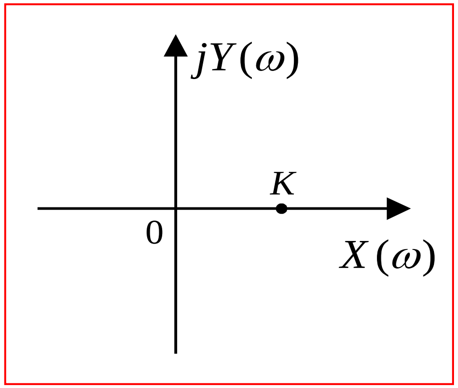
Frequency characteristic $G(j\omega)=K$



It is independent on ω.

The corresponding magnitude and phase characteristics are as follows:

$$\begin{cases} A(\omega)=K \\ \varphi(\omega)=0^\circ \end{cases}$$



$$\begin{cases} L(\omega)=20\lg K \\ \varphi(\omega)=0^\circ \end{cases}$$

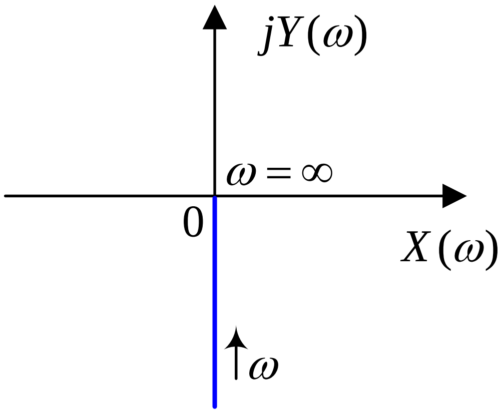
2. Integration element

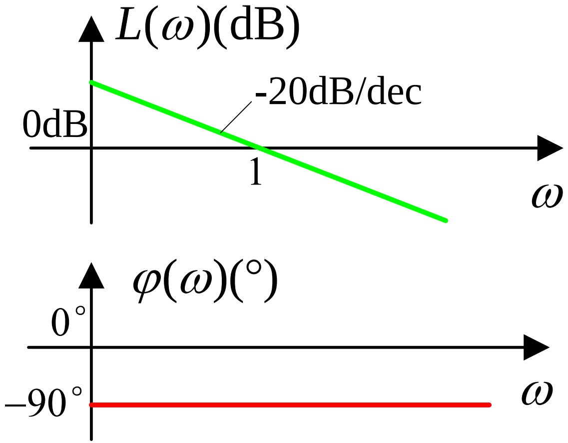
Frequency characteristic

$$G(j\omega)=\frac{1}{j\omega}=\frac{1}{\omega}e^{-j\frac{\pi}{2}}$$

The corresponding magnitude and phase characteristics are as follows:

$$\begin{cases}A(\omega)=1/\omega\\ \varphi(\omega)=-90°\end{cases}\qquad\begin{cases}L(\omega)=-20\lg\omega\\ \varphi(\omega)=-90°\end{cases}$$





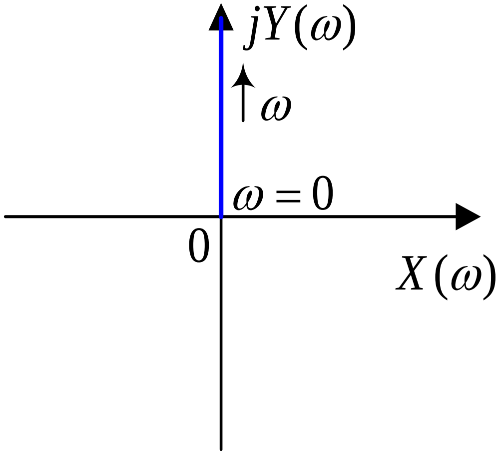
3. Derivative Element

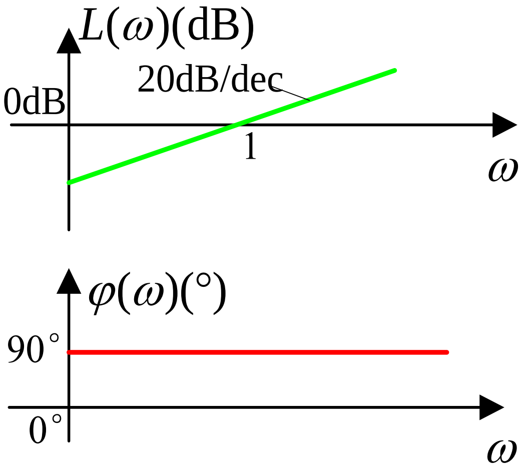
Frequency characteristic

$$G(j\omega) = j\omega = \omega e^{j\frac{\pi}{2}}$$

The corresponding magnitude and phase characteristics are as follows:

$$\begin{cases} A(\omega) = \omega \\ \varphi(\omega) = 90^\circ \end{cases} \qquad \begin{cases} L(\omega) = 20\lg\omega \\ \varphi(\omega) = 90^\circ \end{cases}$$





4.Inertial Element

Frequency characteristic

$$G(j\omega)=\frac{1}{1+j\omega T}$$

Magnitude and phase responses

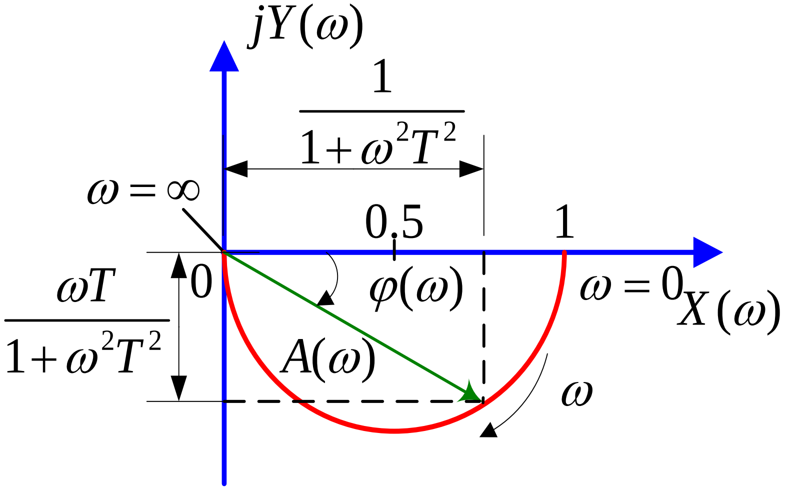
$$\begin{cases} A(\omega)=\dfrac{1}{\sqrt{1+\omega^2T^2}} \\ \varphi(\omega)=-\mathrm{arctg}\,\omega T \end{cases}$$

Rewrite it into real and imaginary parts

$$G(j\omega)=\frac{1}{1+\omega^2T^2}-j\frac{\omega T}{1+\omega^2T^2}=X(\omega)+jY(\omega)$$

$$[X(\omega)-0.5]^2+Y^2(\omega)=0.5^2$$

- **Nyquist diagram is half of the circle with center at (0.5,0) and radius 0.5.**



Log magnitude and phase characteristics are as follows:

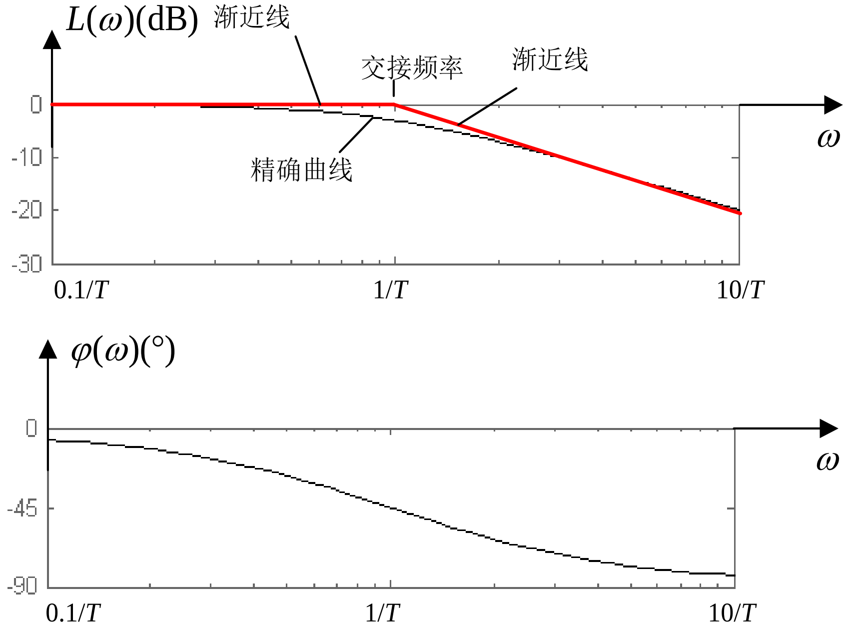
$$\begin{cases} L(\omega) = 20\lg \dfrac{1}{\sqrt{1+\omega^2 T^2}} = -20\lg\sqrt{1+\omega^2 T^2} \\ \varphi(\omega) = -\mathrm{arctg}\,\omega T \end{cases}$$

Low-frequency region： $\omega << 1/T$, $L(\omega) \approx -20\lg 1 = 0$

High-frequency region： $\omega >> 1/T$, $L(\omega) \approx -20\lg \omega T$

Asymptote

The frequency where the low- and high-frequency asymptotes meet is called the break frequency ($\omega = 1/T$).



The true modulus has a value of $L(\omega) = -10\lg(1+1) \approx$ -3dB

- **Remarks:**

① **The error of true modulus and asymptote**

ωT	0.1	0.2	0.5	1	2	5	10
$\Delta L(\omega)$	−0.04	−0.17	−0.97	−3.01	−0.97	−0.17	−0.04

It can be seen that the error at the break frequency is biggest.

② **$\varphi(\omega)$ is symmetrical at all rotations about the point $\omega=1/T, \varphi(\omega)=-45^\circ$**

ωT	0.1	0.2	0.25	0.5	1	2	3	4	5	8	10
$\varphi(\omega)$	–6	–11	–14	–27	–45	–63	–72	–76	–79	–83	–84

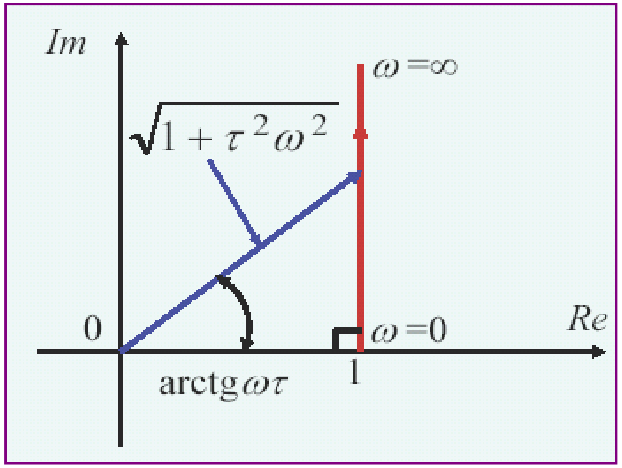
5. First Derivative Element

Frequency characteristic

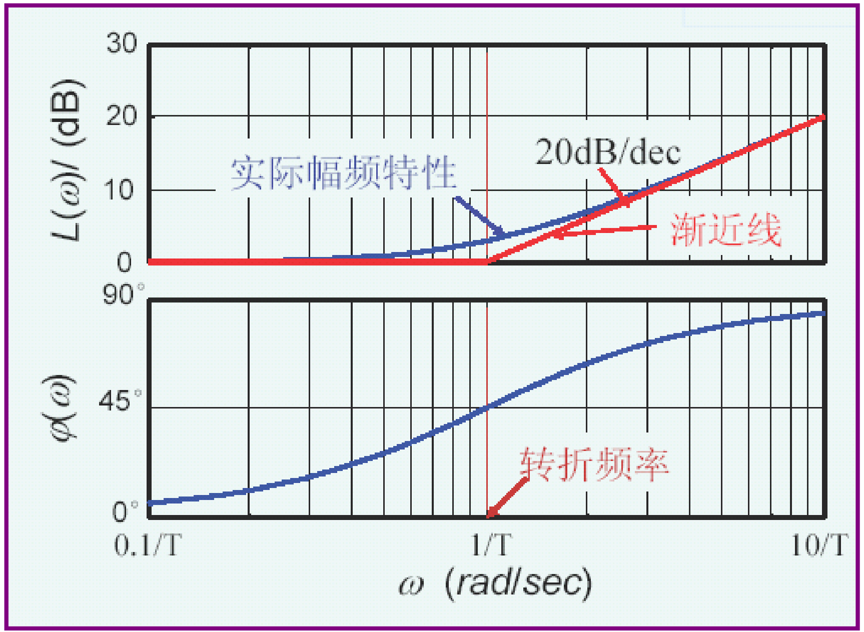
$$G(j\omega)=1+j\omega T$$

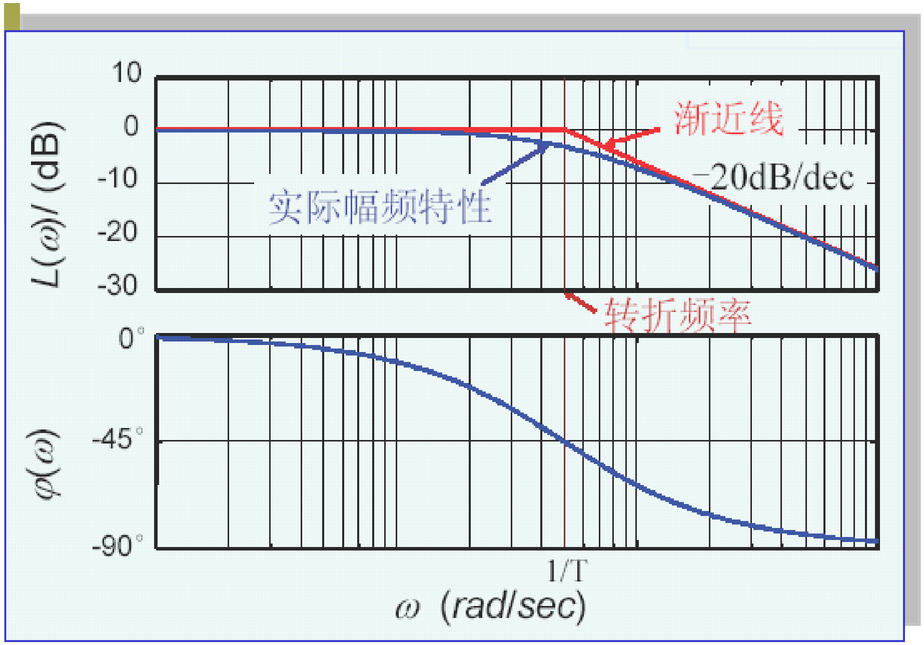
$$\begin{cases} A(\omega)=\sqrt{1+\omega^2T^2} \\ \varphi(\omega)=\operatorname{arctg}\omega T \end{cases}$$

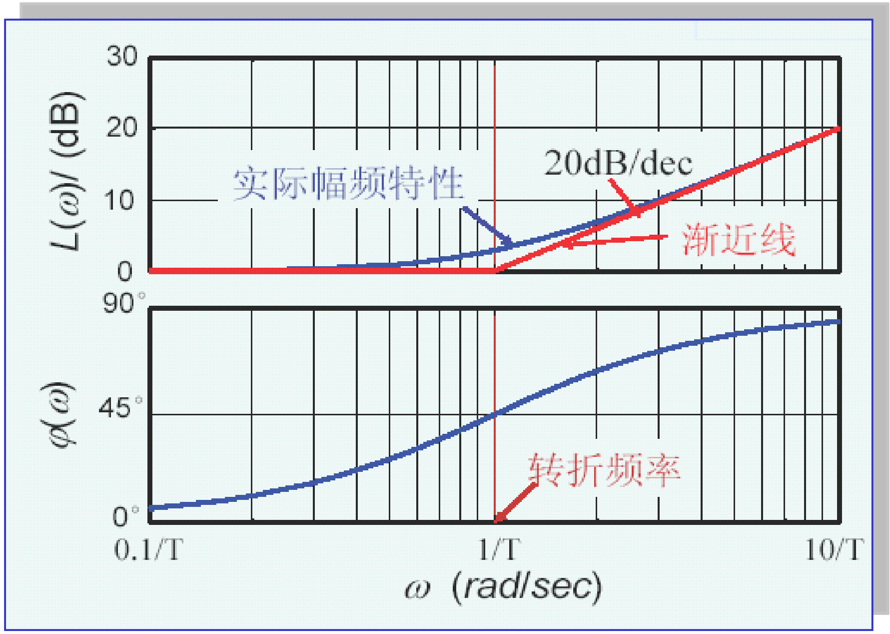
$$\begin{cases} L(\omega)=20\lg\sqrt{1+\omega^2T^2} \\ \varphi(\omega)=\operatorname{arctg}\omega T \end{cases}$$



Nyquist Diagram




Inertial Element


First Derivative Element

Frequency characteristics are the inverse each other

- **Log magnitude characteristic is symmetrical about the line of 0dB**
- **Phase characteristic is symmetrical about 0 degree**

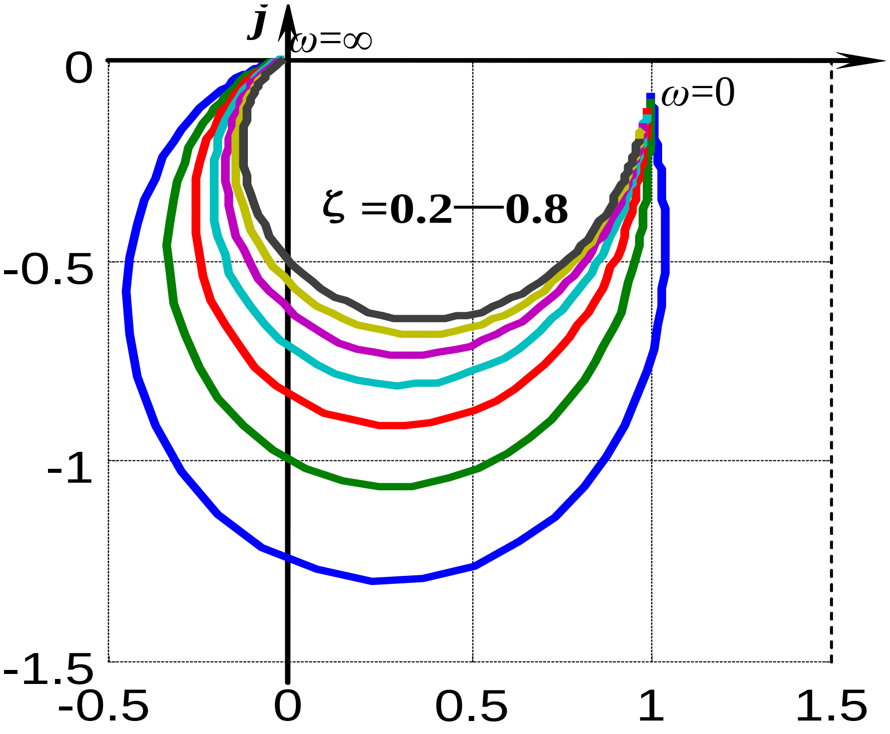
6. Second order oscillation element (Important)

$$G(j\omega)=\frac{1}{1-\frac{\omega^2}{\omega_n^2}+j2\zeta\frac{\omega}{\omega_n}}$$

$$\omega=0,\ G(j0)=1\angle 0^o$$

$$\omega=\omega_n,\ G(j\omega_n)=\frac{1}{2\zeta}\angle -90^o$$

$$\omega=\infty,\ G(j\infty)=0\angle -180^o$$



Nyquist Diagram

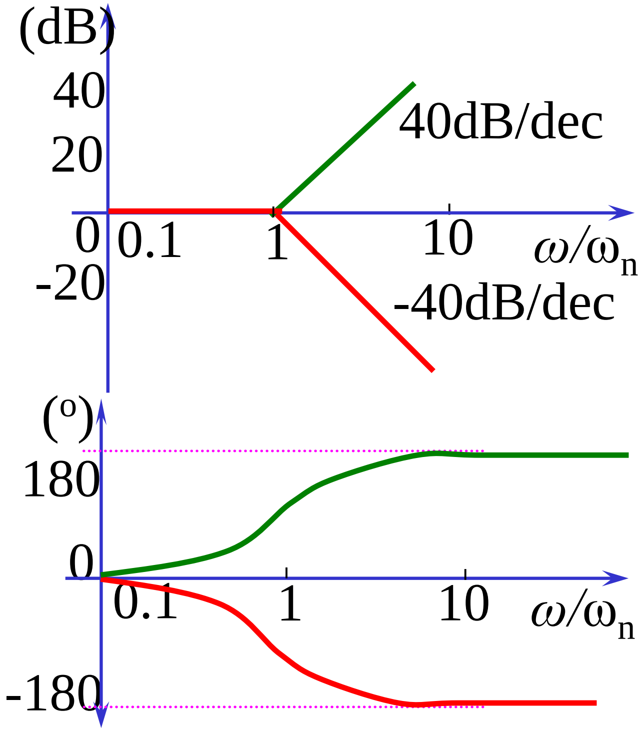
$$L(\omega)=-20\lg\sqrt{(1-\omega^2/\omega_n{}^2)^2+4\zeta^2(\omega/\omega_n)^2}$$

$$\varphi(\omega)=-arctg\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

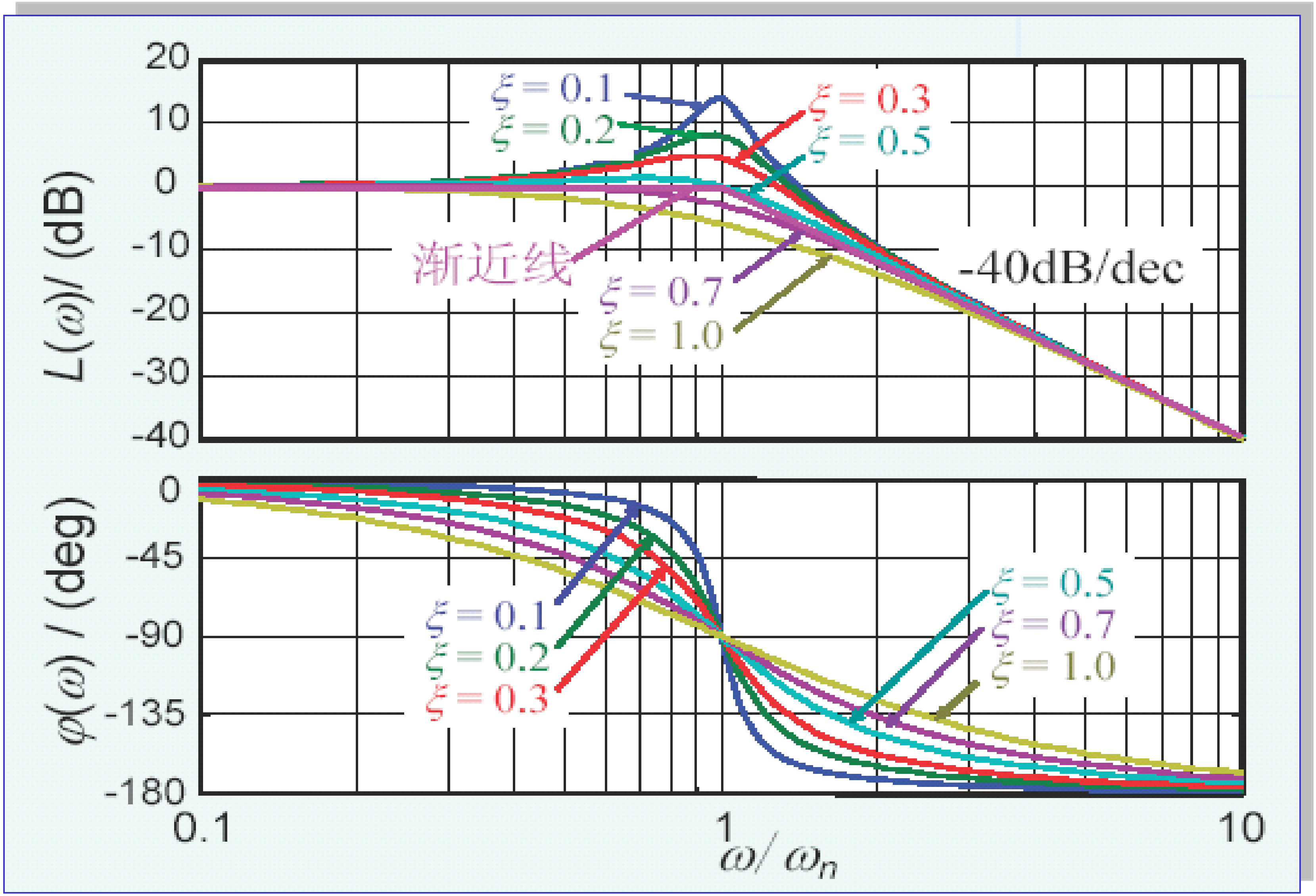
- *For* $\omega<<\omega_n$*,* $L(\omega)\approx 0$
- *For* $\omega>>\omega_n$ *,*

  $L(\omega)\approx -40\lg\omega/\omega_n$

  $=-40(\lg\omega-\lg\omega_n)$



Bode Diagram



Bode Diagram

Remarks:

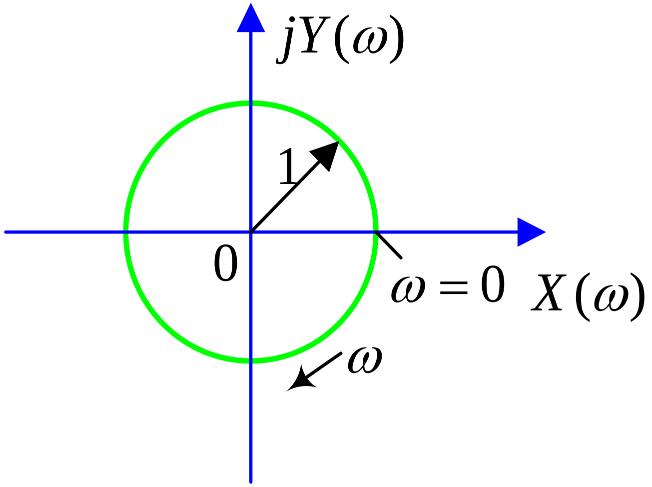
- The low- and high-frequency asymptotes intersect at $\omega = \omega_n$， i.e. the undamped natural frequency。
- Unlike a first-order element which has a single-valued deviation between the approximation and accurate moduli, the discrepancy depends upon the damping ratio ξ.
- The true magnitude may be below or above the straight-line approximate magnitude.
- The resonant peak M_r is the maximum value of $L(\omega)$

7. Delay Element

$$G(j\omega)=e^{-j\omega T}$$

$$\begin{cases} A(\omega)=1 \\ \varphi(\omega)=-\omega T \end{cases}$$

$$\begin{cases} L(\omega)=0 \\ \varphi(\omega)=-\omega T \end{cases}$$



Nyquist Diagram is a circle