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# Chapter 5: Root Locus

# Two Conditions for Plotting Root Locus

**Given open-loop transfer function $G_k(s)$**

**Characteristic equation**

$$G_k(s) = \frac{K_g \prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = -1$$

**Magnitude Condition and Argument Condition**

$$K_g = \frac{\prod_{j=1}^{n}|(s+p_j)|}{\prod_{i=1}^{m}|(s+z_i)|}$$

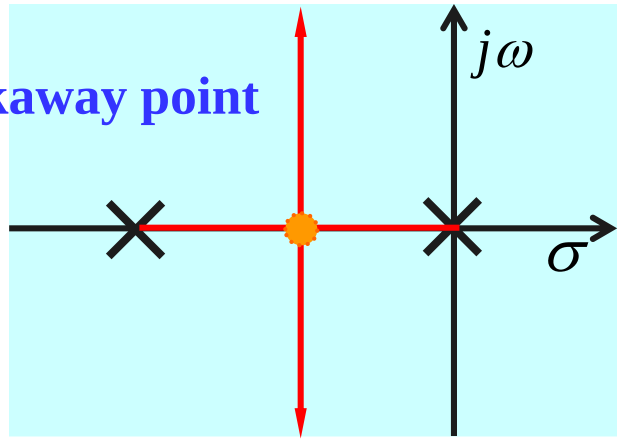
$$\angle\sum_{i=1}^{m}(s+z_i) - \angle\sum_{j=1}^{n}(s+p_j) = \pm(2k+1)\pi, k = 0,1,2..$$

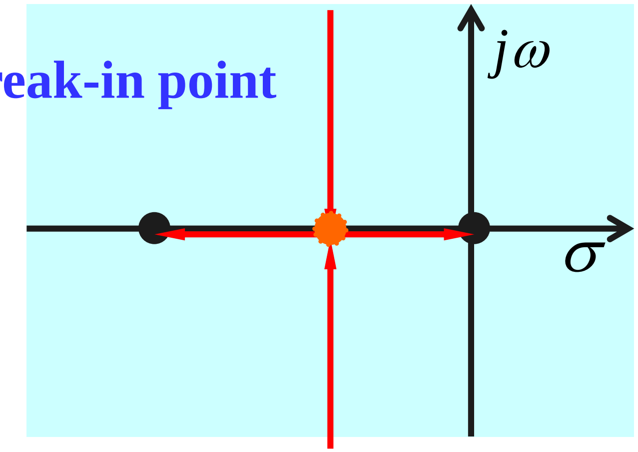
# Rules for Plotting Root Locus

| | Content | Rules |
|---|---|---|
| 1 | Continuity and Symmetry | Symmetry Rule |
| 2 | Starting and end points<br>Number of segments | *n* segments start from n open-loop poles, and end at *m* open-loop zeros and *(n-m)* zeros at infinity. |
| 3 | Segments on real axis | On the left of an odd number of poles or zeros |
| 4 | Asymptote | *n-m* segments: $\alpha = \frac{(2k+1)}{n-m}\pi, k = 0, \pm 1, \pm 2, \ldots$ |
| 5 | Asymptote | $\sigma = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{m}(-z_i)}{n-m}$ |

| | | |
|---|---|---|
| **6** | **Breakaway and break-in points** | $\dfrac{d[F(s)]}{ds} = 0 \quad F(s) = P(s) + K_g Z(s) = 0$<br>$P(s)\dot{Z}(s) - \dot{P}(s)Z(s) = 0$<br>$\sum_{i=1}^{m} \dfrac{1}{z_i - \delta} = \sum_{j=1}^{n} \dfrac{1}{p_i - \delta}$ |
| **7** | **Angle of emergence and entry** | **Angle of emergence**<br>$\varphi_p = \mu\pi(2k+1) + \sum_{i=1}^{m} \theta_i - \sum_{\substack{j=1 \\ j \neq p}}^{n} \varphi_j$<br>**Angle of entry**<br>$\theta_z = \pm\pi(2k+1) + \sum_{j=1}^{n} \varphi_j - \sum_{\substack{i=1 \\ i \neq z}}^{m} \theta_i$ |
| **8** | **Cross on the imaginary axis** | **Substitute $s = j\omega$ to characteristic equation and solve**<br>**Routh's formula** |

# Rule 6: Breakaway and Break-in Points on the Real Axis




**Use the following necessary condition**

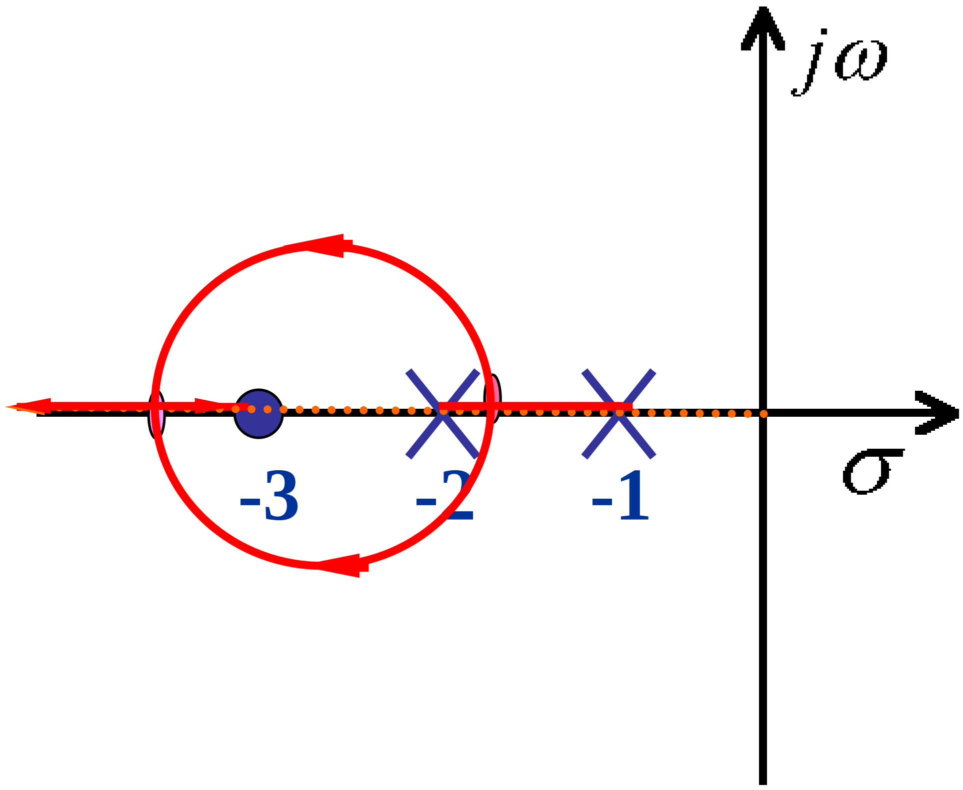
$$\frac{\mathrm{d}\left[G(s)H(s)\right]}{\mathrm{d}s}=0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{G(s)H(s)}\right]=0 \quad \text{or} \quad \frac{\mathrm{d}K_g}{\mathrm{d}s}=0$$

$$P(s)\dot{Z}(s)-\dot{P}(s)Z(s)=0$$

$$\sum\frac{1}{s+z_i}=\sum\frac{1}{s+p_j}$$

**Example 5.3.1: Given the open-loop transfer function, please draw the root locus.**

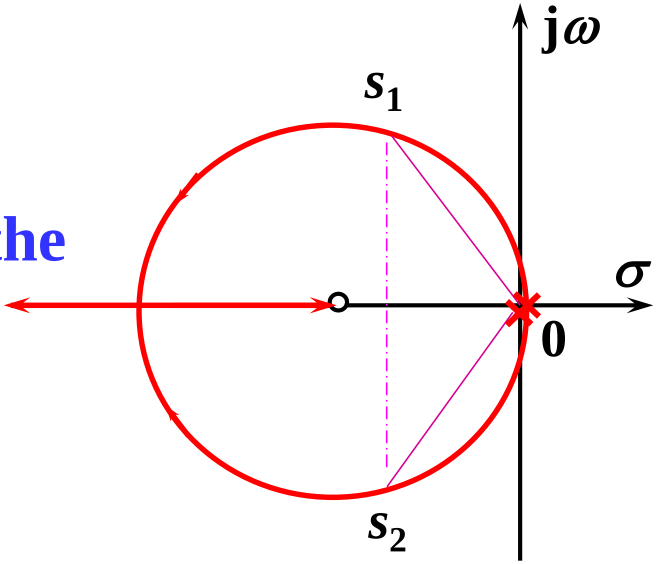
$$G(s)H(s)=\frac{k(s+3)}{(s+1)(s+2)}$$



## Example 5.3.2: Given the open-loop transfer function

$$G_k(s) = \frac{K^*(s+1)}{s^2}$$

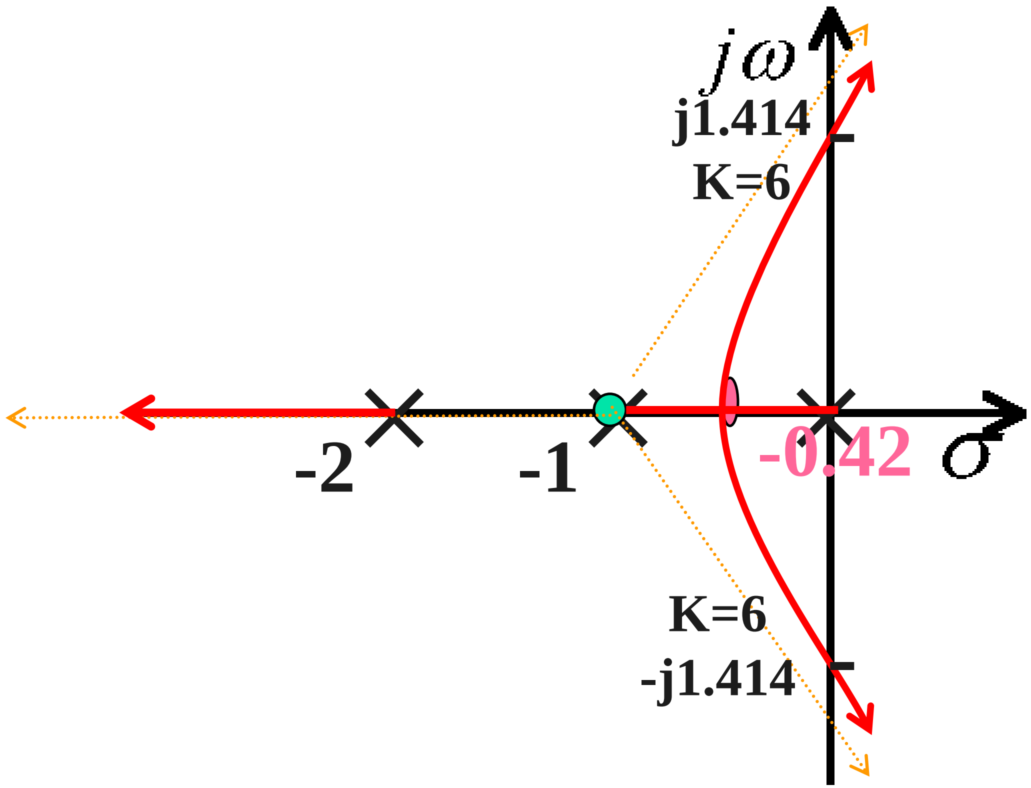
please prove that the root locus in the complex plane is a circle.




**Conclusion:** For the open-loop transfer function with one zero and two poles, the root locus of characteristic equation is probably a circle in the complex plane.

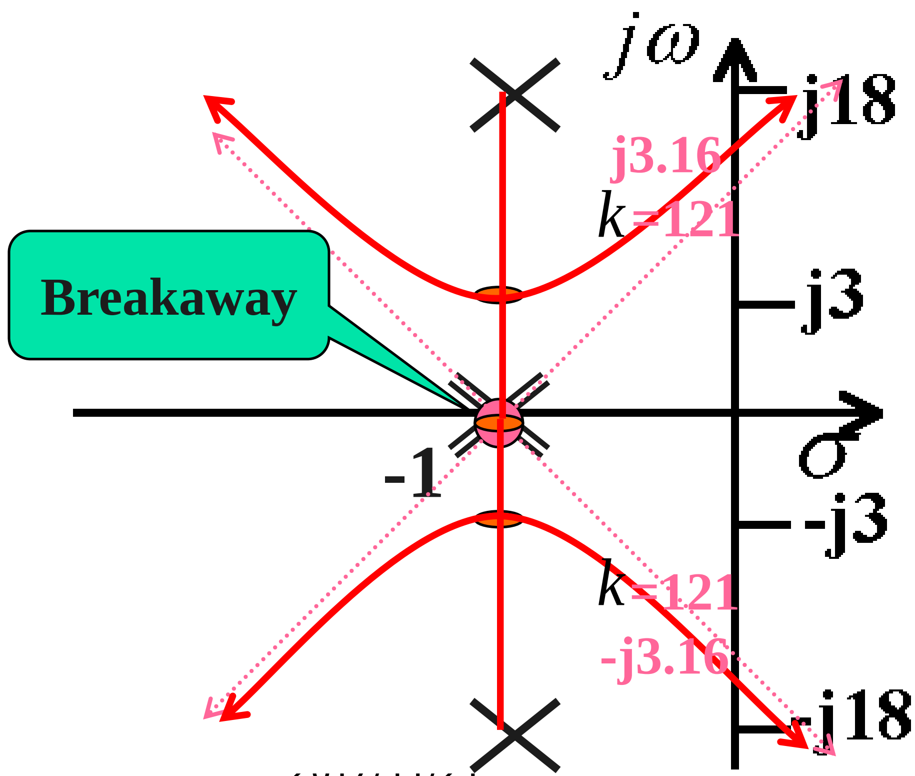
## Example 5.3.3:

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

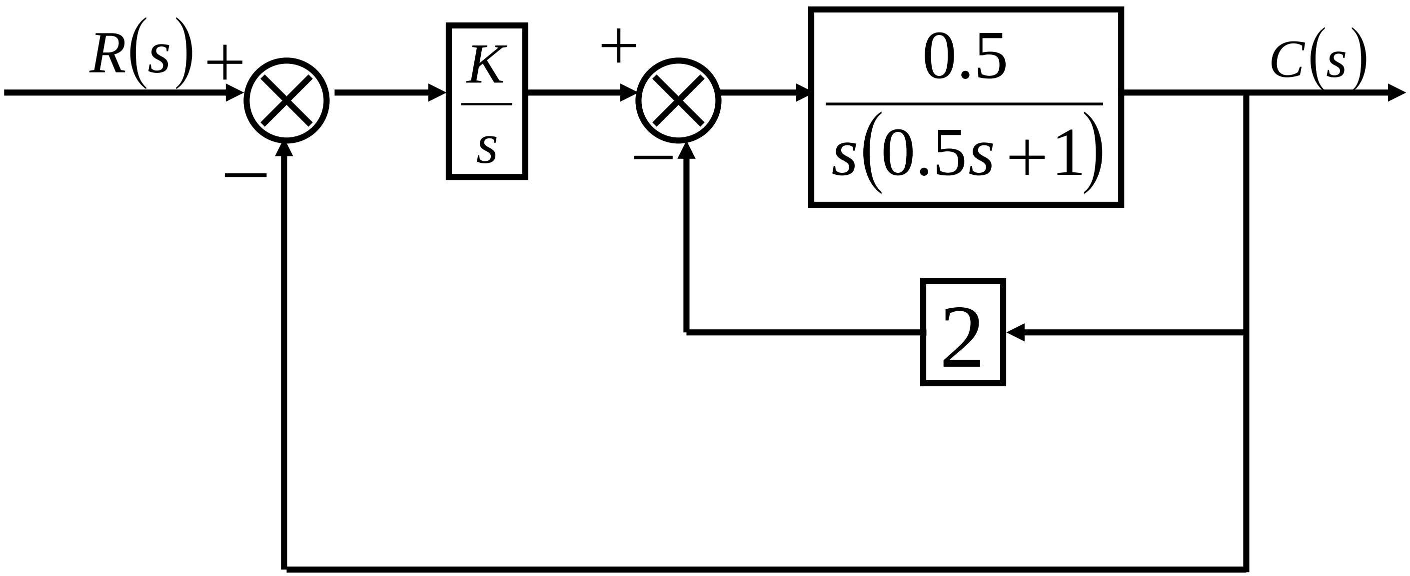


# Example 5.3.4:

$$G(s)H(s) = \frac{k}{(s+1)^2(s+1+j\sqrt{18})\,(s+1-j\sqrt{18})}$$



# Example 5.3.5




**Please sketch the root locus with respect to *K*=*[0*,+∞*)*.**

# Extension of Root Locus

**Canonical form**

$$G_k(s) = K_g \cdot \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = -1$$

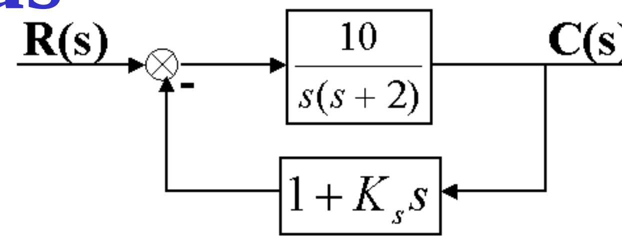
**Conventional Root Locus**

**Root locus gain** → $K_g$

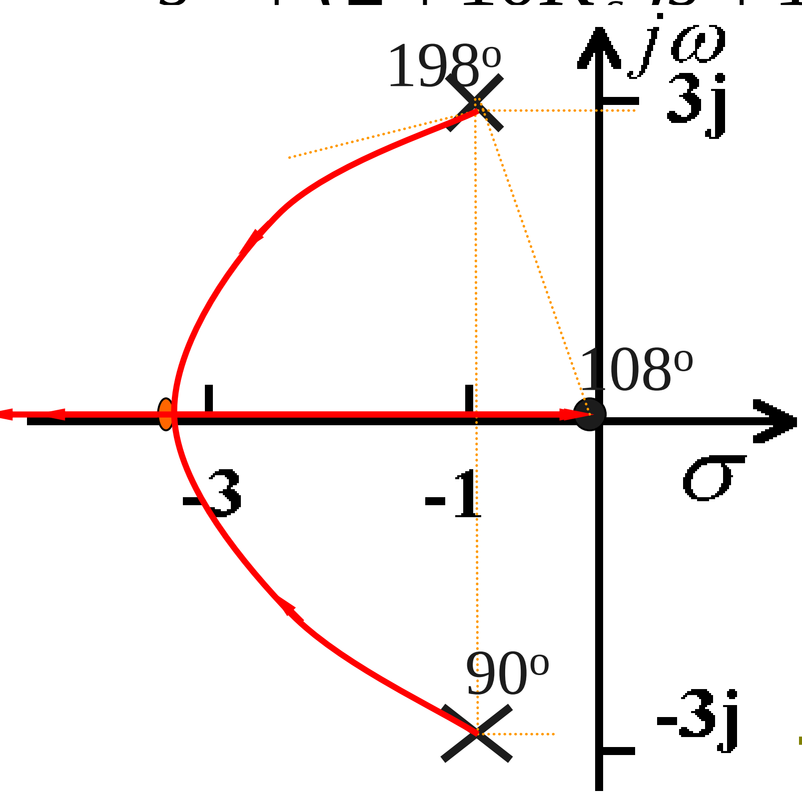
1. **How to sketch the root locus for other parameters?** ⇒ ■ **Parameter Root Locus**
2. **How to sketch if** $G_k(s)=1$ ⇒ ■ **Zero-degree Root Locus**

# 5.3.2 Parameter Root Locus

**Example 5.3.6:** $K_s$ **is a ramp feedback gain, please sketch the root locus with respect to** $K_s = [0, +\infty)$**.**




$$s^2 + (2 + 10K_s)s + 10 = 0$$




$$s^2 + 2s + 10 + 10K_s s = 0$$

$$\frac{10K_s s}{s^2 + 2s + 10} + 1 = 0$$

$$1 + G'(s)H'(s) = 0$$

$$G'(s)H'(s) = \frac{10K_s s}{s^2 + 2s + 10}$$

$$\varphi = 180^\circ - (90^\circ - 108^\circ) = 198^\circ$$

# 5.3.2 Zero-degree Root Locus

**Example: Sketch the root loci of the system with the open loop transfer function:**

$$G(s)H(s)=\frac{K_1(1-s)}{s(s+1)} \qquad K_1: 0 \to +\infty$$

**Analysis:** $G(s)H(s)=\frac{K_1(1-s)}{s(s+1)}=\frac{-K_1(s-1)}{s(s+1)}$

**For this kind of systems, the characteristic equations are like as:**

$$1-K_1G_1(s)H_1(s)=0 \Rightarrow K_1G_1(s)H_1(s)=1$$

**Magnitude equation** $|G_k(s)|=\frac{1}{K_g} \quad K_g: 0 \to +\infty$

**Argument equation** $\angle G_k(s)=\pm 2k\pi, k=0,\pm 1,\pm 2,\cdots$

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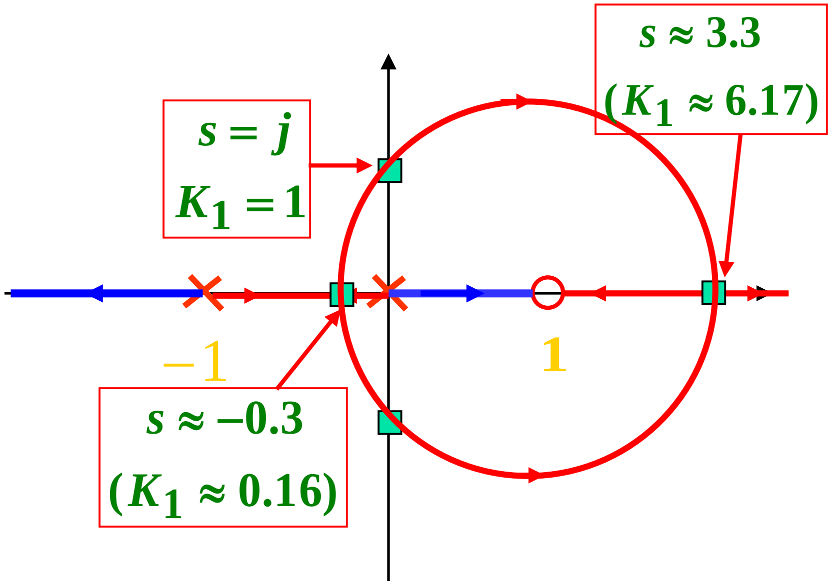
Root locus by using the sketching rules with the following modification:

- Real-axis: Left of even number of zeros or poles
- Asymptote
- Angles of emergence and entry

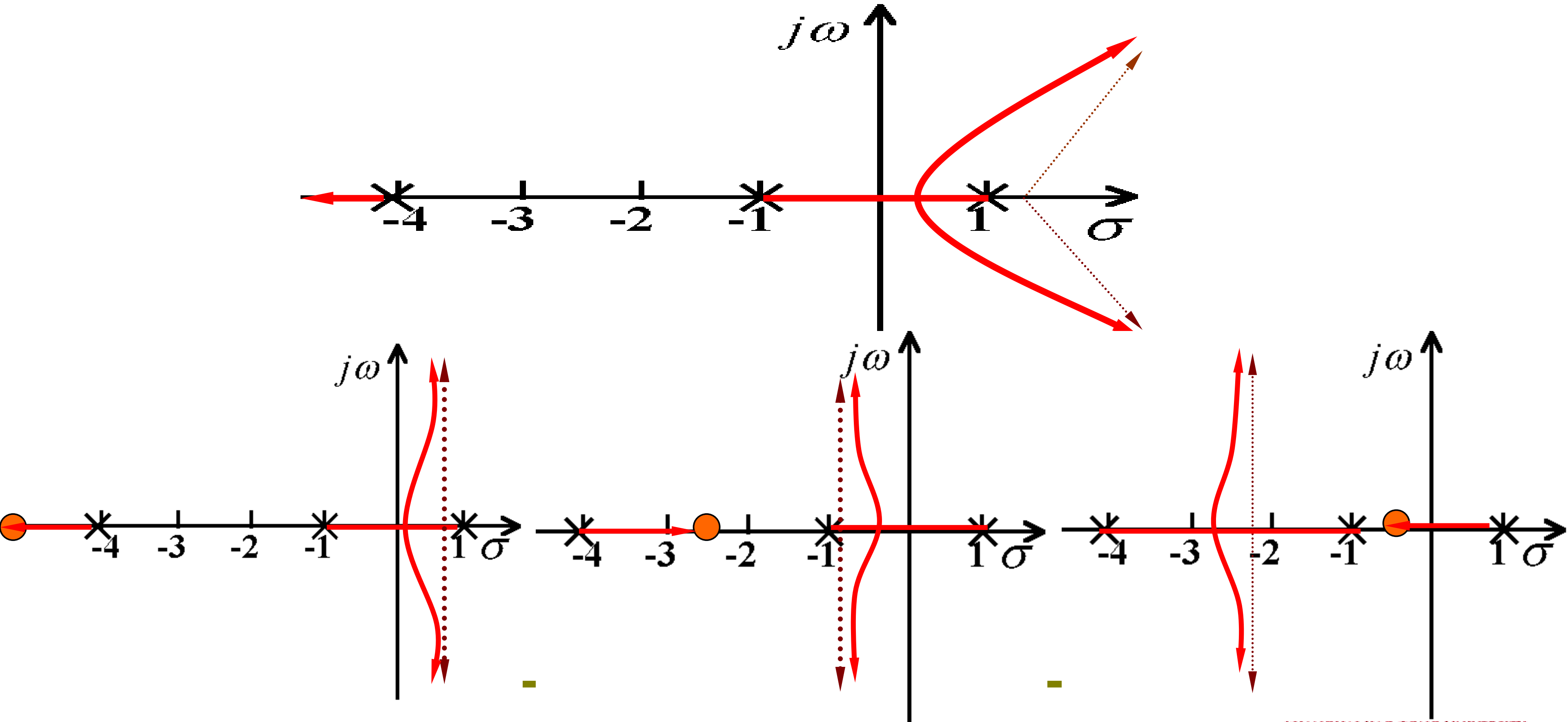
$$\varphi_p = \mu 2k\pi + \sum_{i=1}^{m}\theta_i - \sum_{\substack{j=1 \\ j\neq p}}^{n}\varphi_j$$




For $K_g$ varying from $-\infty \rightarrow 0$ together with $K_g = [0, +\infty)$ simultaneously, the root loci are named the "complete root loci".

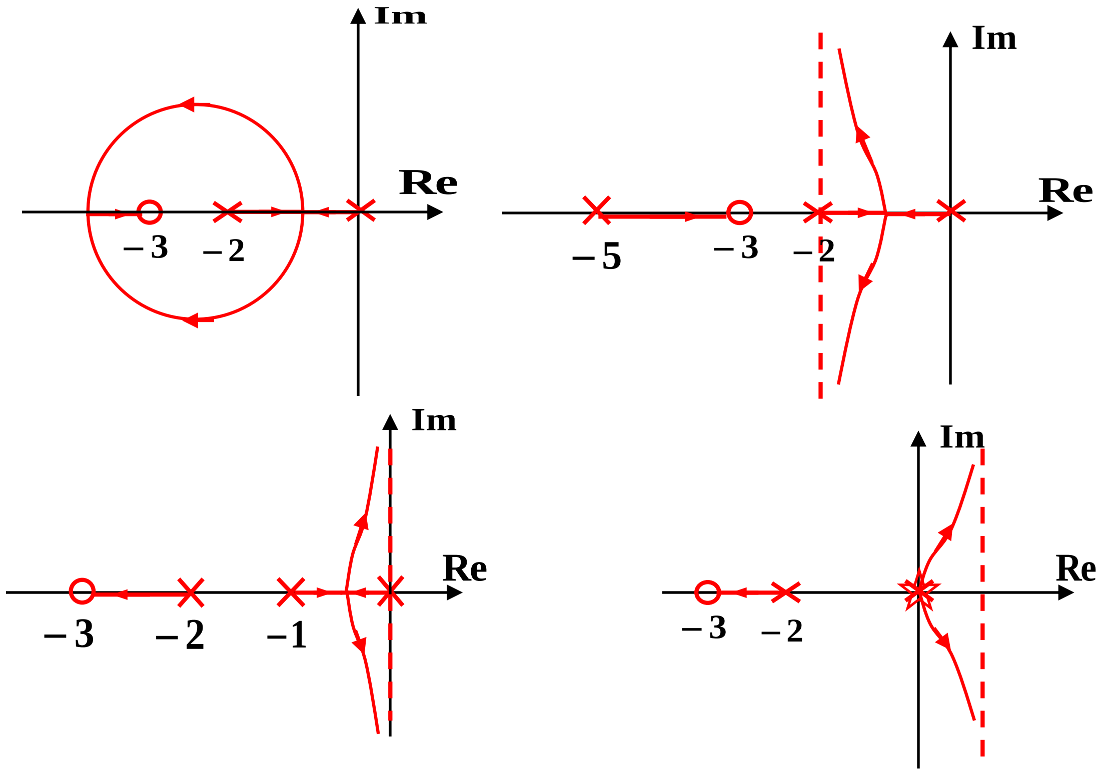
# 5-4 Application of Root Locus

**Insert a zero** $G_k(s)=\dfrac{K_g}{(s-1)(s+1)(s+4)} \Rightarrow G'_k(s)=\dfrac{K_g(s+a)}{(s-1)(s+1)(s+4)}$



# Add a pole to the open-loop transfer function

$$GH_1(s)=\frac{K_1(s+3)}{s(s+2)} \Rightarrow GH_2(s)=\frac{K_1(s+3)}{s(s+2)(s+a)}$$



# 5.4.1 The effects of Zeros and Poles

## Attracting effect

Generally, adding an open zero in the left *s*-plane will lead the root loci to be bended to the left.

The more closer to the imaginary axis the open zero is, the more prominent the effect on the system's performance is.

## Repelling effect

Generally, adding an open pole in the left *s*-plane will lead the root loci to be bended to the right.

The more closer to the imaginary axis the open pole is, the more prominent the effect on the system's performance is.

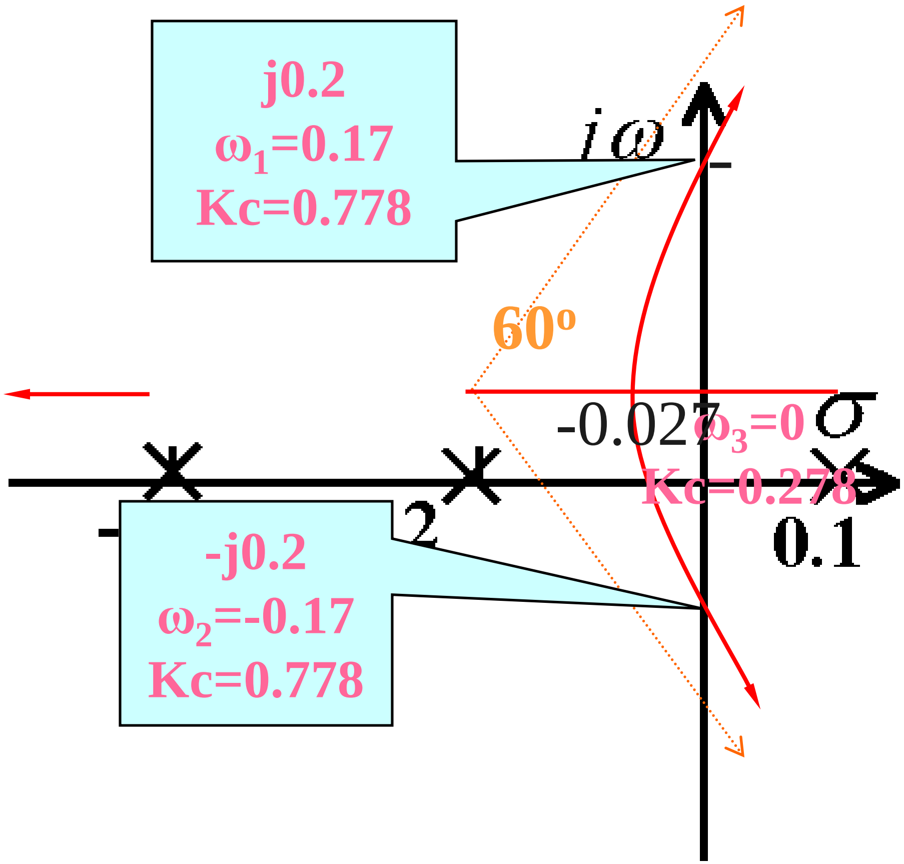
**Example 5.4.1:**

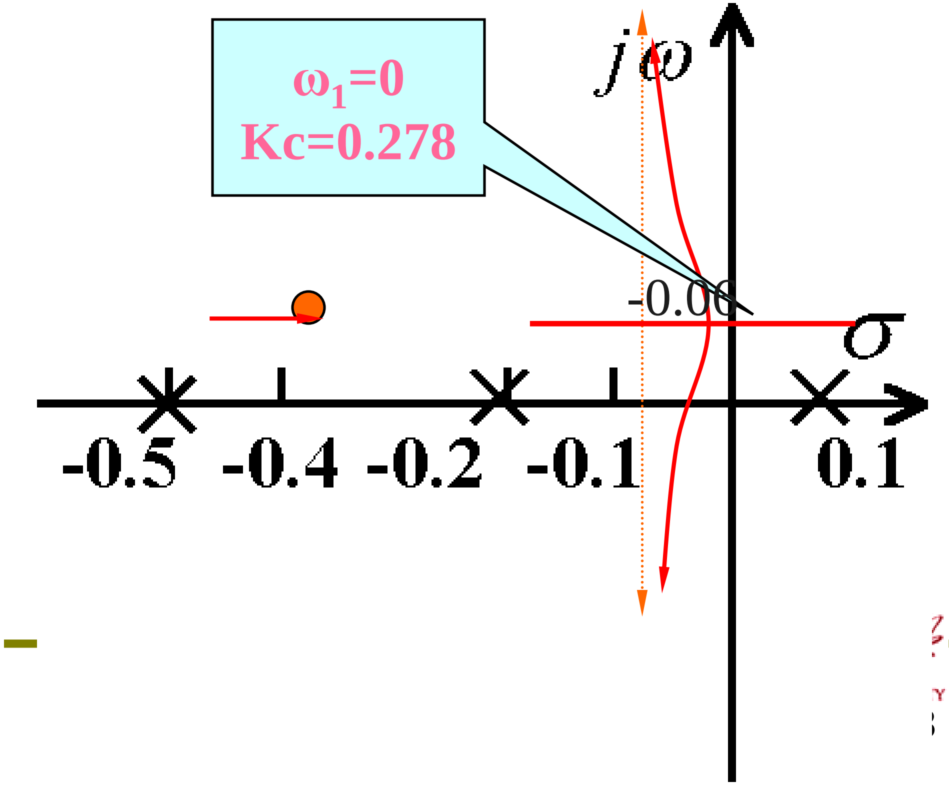
**How to enlarge the stability region?**

$$G_k(s) = \frac{3.6K_c}{(10s-1)(5s+1)(2s+1)}$$

$$G_k(s) = \frac{0.036K_c}{(s-0.1)(s+0.2)(s+0.5)} = \frac{K}{(s-0.1)(s+0.2)(s+0.5)}$$

$$G'_k(s) = \frac{K(s+0.4)}{(s-0.1)(s+0.2)(s+0.5)}$$



# 5.4.2 Performance Analysis Based on Root Locus

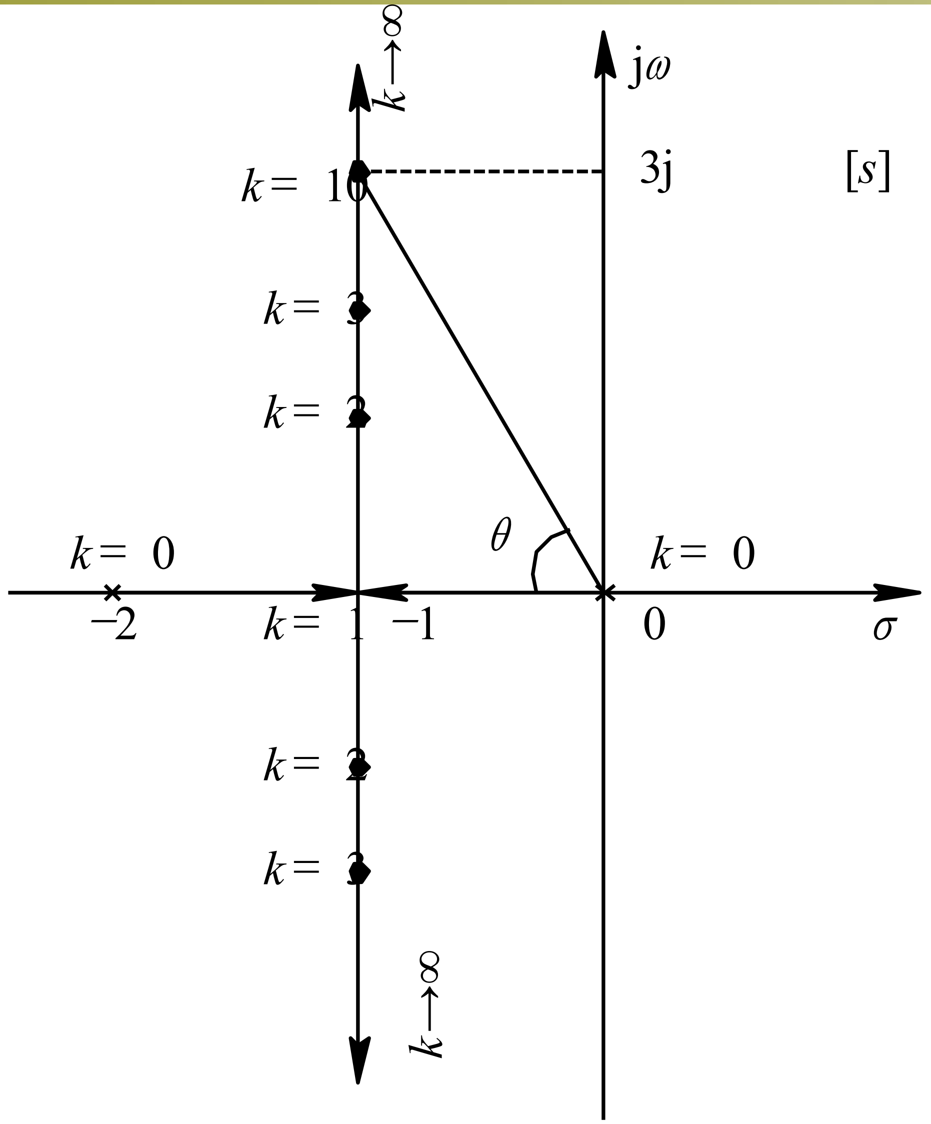
**Example 5.4.2: Given the open-loop transfer function**

$$G_k(s) = \frac{K}{s(0.5s+1)}$$

**please analyze the effect of open-loop gain *K* on the system performance.**

**Calculate the dynamic performance criteria for *K*＝*5*.**

■



k→∞
jω
k = 10
3j
[s]
k = 3
k = 2
θ
k = 0
k = 0
−2
k = 1
−1
0
σ
k = 2
k = 3
k→∞

**It is observed from the root locus that the system is stable for any *K*.**

**For 0＜*K*＜0.5(0＜*k*＜1), there are two different negative real roots.**

**For *K*=0.5(*k*=1), there are two same negative real roots. For *K*＞0.5(*k*＞1), there are a pair of conjugate complex poles.**

For *K*=5( *k*=10), the closed-loop poles are

$$s_{12} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = -1 \pm j3$$

$$\omega_n = \sqrt{10} = 3.16, \xi = \frac{1}{3.16} = 0.316$$

**The criteria for transient performance can be given by**

**Overshoot**

$$\sigma_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100\% = e^{-1.05} \times 100\% = 35\%$$

**Peak time** $$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 1.05s$$

**Settling time** $$t_s = \frac{3}{\xi\omega_n} = 3s \quad (\Delta = \pm 5\%)$$

**1. We can get the information of the system's stability in terms of that whether the root loci always are in the left-hand s-plane with the system's parameter varying.**

**2. We can get some information of the system's steady-state error in terms of the number of the open-loop poles at the origin of the s-plane.**

**3. We can get some information of the system's transient performance in terms of the tendencies of the root loci with the system's parameter varying.**

**4. The root loci in the left-hand s-plane move to far from the imaginary axis with the system's parameter varying, the system's response is more rapid and the system is more stable, vice versa.**

# Complement

- Use Matlab to sketch root locus

$$G_k(s)=\frac{K_g\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=\frac{K_g(s+3)(s+4)}{(s+1)(s+2)}=\frac{K_g(s^2+7s+12)}{s^2+3s+2}$$

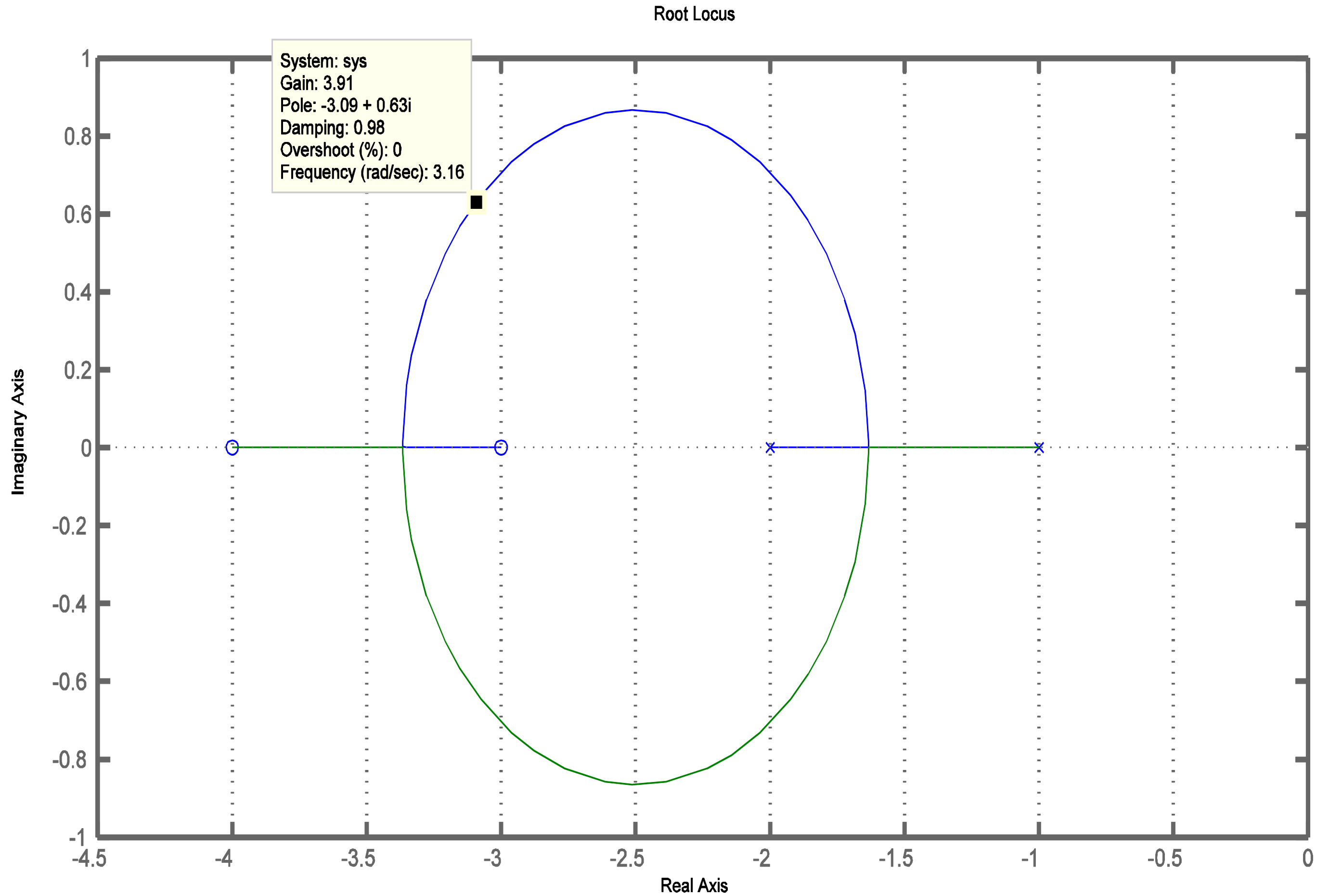
**In Matlab:**

**num=[1 7 12]**

**den=[1 3 2]**

} **sys=tf[num,den]**

**Rlocus(num,den)**


Root Locus
System: sys
Gain: 3.91
Pole: -3.09 + 0.63i
Damping: 0.98
Overshoot (%): 0
Frequency (rad/sec): 3.16
Imaginary Axis
Real Axis
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-4.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0