上海交通大學

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Chapter 5: Root Locus

Key conditions for Plotting Root Locus

Given open-loop transfer function $G_k(s)$

Characteristic equation

$$G_k(s) = \frac{K_g \prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = -1$$

Magnitude Condition and Argument Condition

$$K_g = \frac{\prod_{j=1}^{n}|(s+p_j)|}{\prod_{i=1}^{m}|(s+z_i)|}$$

$$\angle\sum_{i=1}^{m}(s+z_i) - \angle\sum_{j=1}^{n}(s+p_j) = \pm(2k+1)\pi, k = 0,1,2..$$

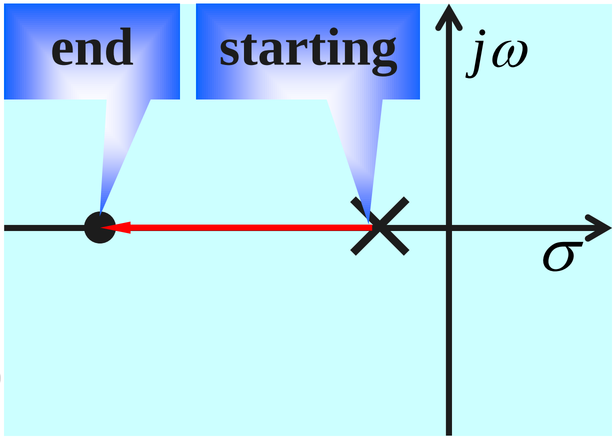
5-3 Rules for Plotting Root Locus

5.3.1 Rules

Rule 1: Starting and end points

1)For K_g=0, we can get from magnitude equation that

$$K_g = \frac{|s+p_1|\ldots|s+p_n|}{|s+z_1|\ldots|s+z_m|} \qquad s=-p_j \quad (j=1,2\ldots n)$$



2) For $K_g \to +\infty$, it may result in one of the following facts

$$s=-z_i \qquad (i=1,2\ldots m)$$

$$s\to\infty \qquad \text{(n>m)}$$

Using Magnitude Equation

Rule #1: The locus starts at a pole for $K_g=0$ and finishes at a zero or infinity when $K_g=+\infty$.

Poles and zeros at infinity

$G_k(s)$* has a zero at infinity if *$G_k(s \to +\infty) \to 0$

$G_k(s)$* has a pole at infinity if *$G_k(s \to +\infty) \to +\infty$

Example

$$G_k(s) = \frac{K}{s(s+1)(s+2)}$$

This open-loop transfer function has three poles, 0,-1,-2. It has no finite zeros.

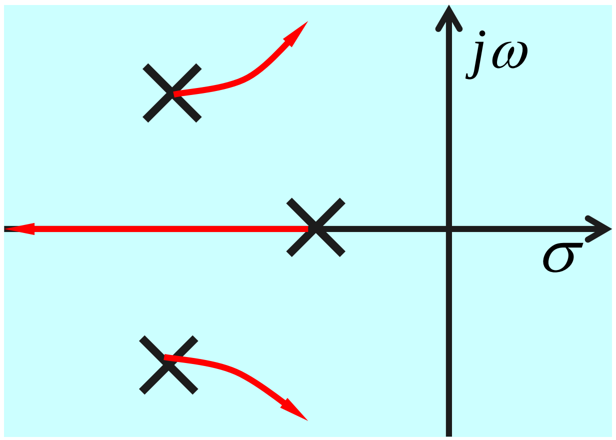
For large *s*, we can see that $G_k(s) \approx \frac{K}{s^3}$ **.**

So this open-loop transfer function has three *(n-m)* zeros at infinity.

Rule 2: Number of segments

Rule #2: The number of segments equals to the number of poles of open-loop transfer function.

m **segments end at the zeros, and *(n-m)* segments goes to infinity.**



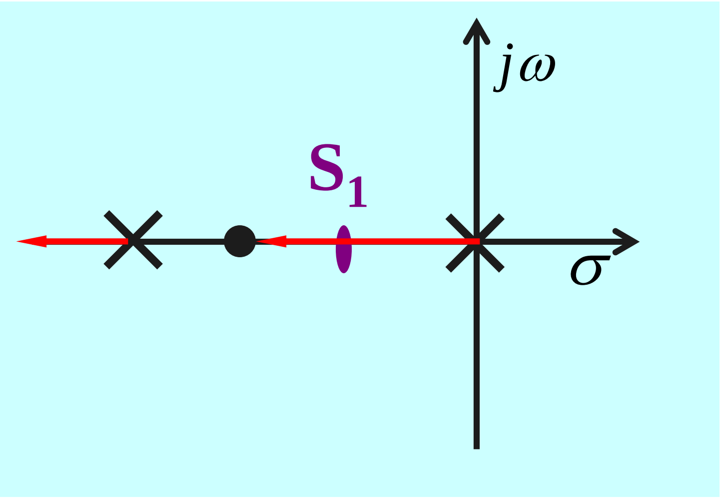
Sometimes, max{m,n}

Rule 3: Symmetry rule

Rule #3: The loci are symmetrical about the real axis since complex roots are always in conjugate pairs.

Rule 4: Segments of the real axis

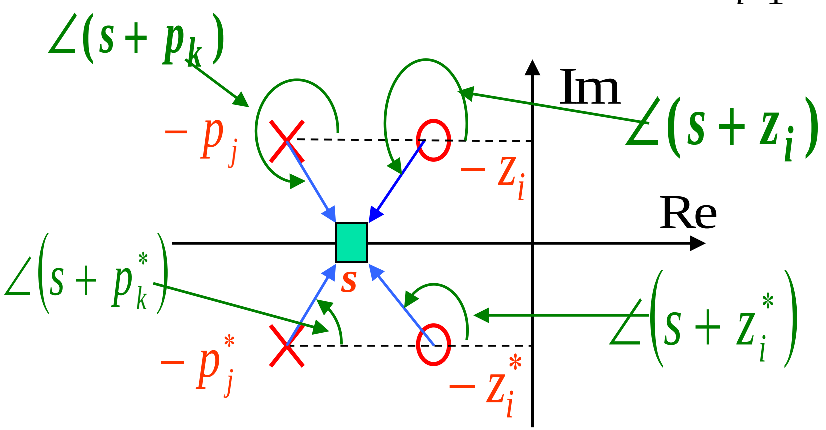
Segments of the real axis to the left of an odd number of poles and zeros are segments of the root locus, remembering that complex poles or zeros have no effect.

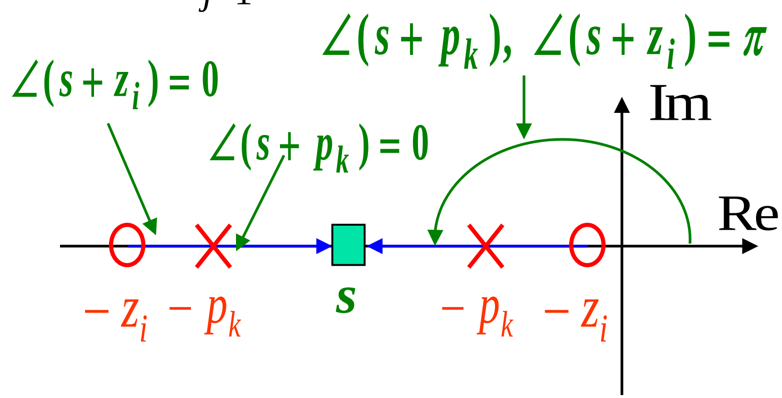


Using Argument Equation

Example Argument equation

$$\angle G_k(s) = \pm(2k+1)\pi \Rightarrow \sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j) = \pm(2k+1)$$





For complex zeros and poles

$$\angle(s+p_k)+\angle(s+p_k^*)=360^\circ$$

$$\angle(s+z_i)+\angle(s+z_i^*)=360^\circ$$

For real zeros and poles on the right

$$\angle(s+p_k)=\pi$$

$$\angle(s+z_i)=\pi$$

Real-axis segments are to the left of an odd number of real-axis finite poles/zeros.

5. Asymptotes of locus as *s* Approaches infinity

The asymptotes intersect the real axis at σ, where

$$\sigma = \frac{(-p_1 - p_2 - \cdots - p_n) - (-z_1 - z_2 - \cdots - z_m)}{n-m}$$

$$= \frac{\sum_{j=1}^{n}\left(-p_j\right) - \sum_{i=1}^{m}\left(-z_i\right)}{n-m}$$

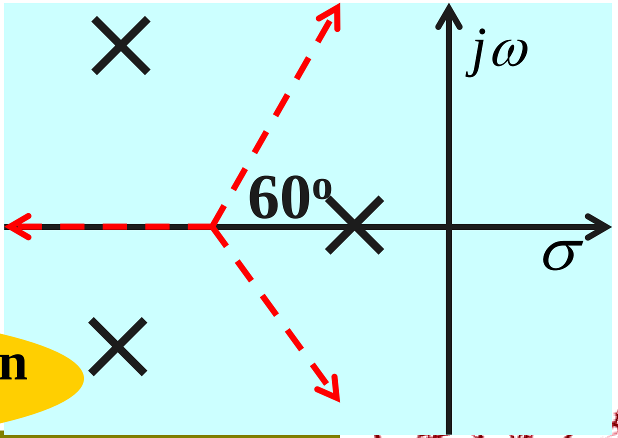
The intercept σ can be obtained by applying the theory of equations.

The angle between asymptote and positive real axis is

$$\phi = \frac{\pm 180^{o}(2k+1)}{n-m} \qquad (k = 0, 1, 2, \cdots)$$

To obey the symmetry rules, the negative real axis is one asymptote when *n-m* is odd.

Using Argument Equation



Example

$$G_k(s) = \frac{K}{s(s+1)(s+2)}$$

This open-loop transfer function has three finite poles and three zeros at infinity.

***(n-m)* segments go to zeros at infinity.**

$$|G_k(s)| = \left|\frac{1}{s(s+1)(s+2)}\right| = \frac{1}{K_g} \to 0, K_g \to +\infty$$

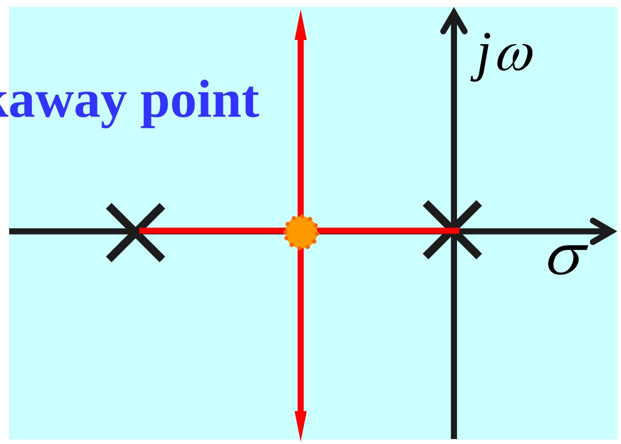
$$\approx \frac{1}{s^3} \to 0 \qquad \phi = 60^{\circ}, -60^{\circ}, 180^{\circ}$$

Assume the root of closed-loop system s_1 at infinity has the same angle to each finite zero or pole.

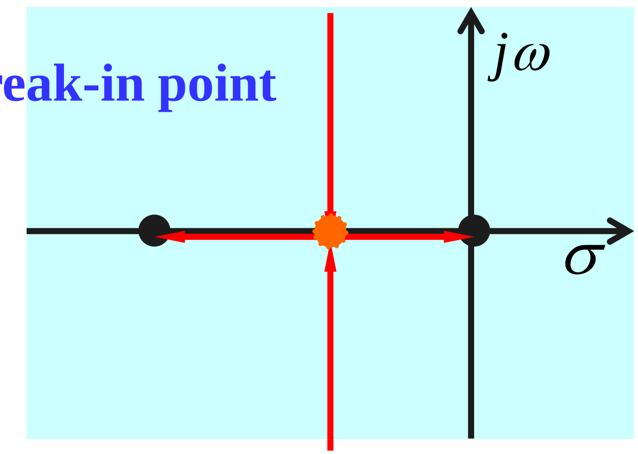
$$(n-m)\phi = \pm 180^{\circ}(2k+1) \qquad k = 0, 1, 2, \ldots$$

Rule 6: Breakaway and Break-in Points on the Real Axis

Breakaway point



Break-in point



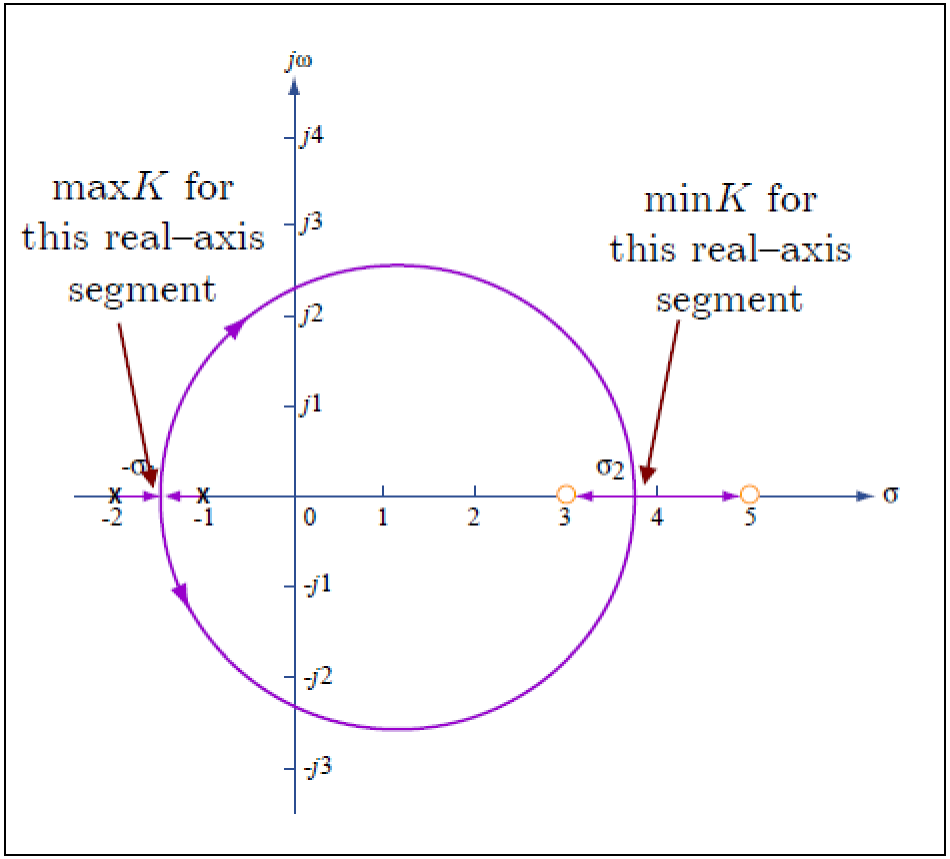
When the root locus has segments on the real axis between two poles, there must be a point at which the two segments break away from the real axis and enter the complex region.

For two finite zeros or one finite zero and one at infinity, the segments are coming from complex region and enter the real axis.

Using Magnitude Equation

Breakaway point

K_g starts with zero at the poles.
There is a point somewhere the K_g for the two segments simultaneously reach a maximum value.



Break-in point

The break-in point is that the value of K_g is a minimum between two zeros.

How?

- **Express K_g as a function of s**
- **Differentiating the function with respect to s equals to zero and solve for s**

Characteristic equation

$$1+G_k(s)=1+\frac{K_g Z(s)}{P(s)}=\frac{F(s)}{P(s)}=0 \quad\Longrightarrow\quad K_g=-\frac{P(s)}{Z(s)}$$

Assuming there are r repeated roots at the point *S_1*, *F(s)* can be rewritten into

$$F(s)=P(s)+K_g Z(s)$$

$$=(s-s_1)^r(s-s_2)\Lambda\,(s-s_{n-r})$$

$$\left.\frac{dF(s)}{ds}\right|_{s=s_1}=\frac{dP(s)}{ds}+K_g\frac{dZ(s)}{ds}=0$$

$$\Longrightarrow\quad K_g=-\frac{\dot{P}(s)}{\dot{Z}(s)}=-\frac{P(s)}{Z(s)}$$

With the solution of *s*, we can get *K_g*.

For positive *K_g*, the corresponding point may be the breakaway or break-in point.

Use the following necessary condition

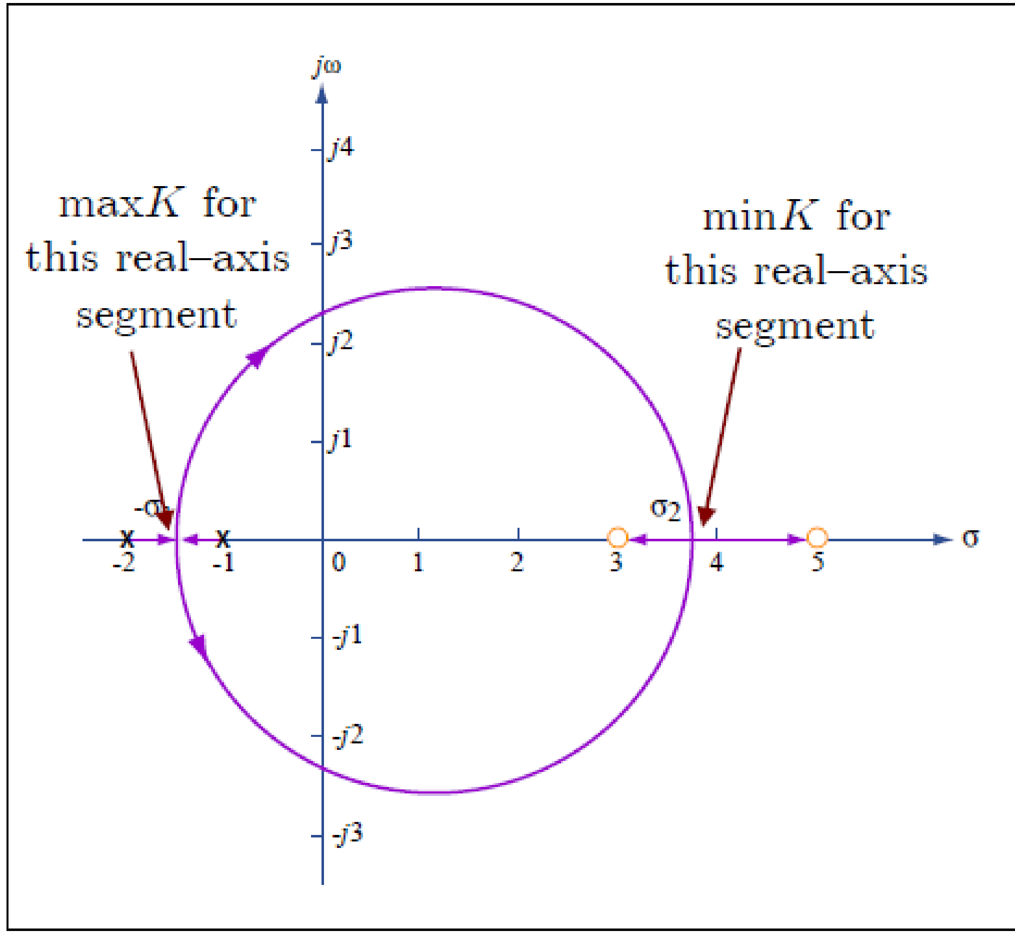
$$P(s)\dot{Z}(s)-\dot{P}(s)Z(s)=0$$

Example

$$G_k(s)=K_g\frac{(s-3)(s-5)}{(s+1)(s+2)}=K_g\frac{Z(s)}{P(s)}$$

$$P(s)\dot{Z}(s)-\dot{P}(s)Z(s)$$
$$=11s^2-26s-61=0$$

$$s_1=-1.45,\ s_2=3.82$$



Alternatively, we can solve

$$\sum\frac{1}{s+z_i}=\sum\frac{1}{s+p_j}$$

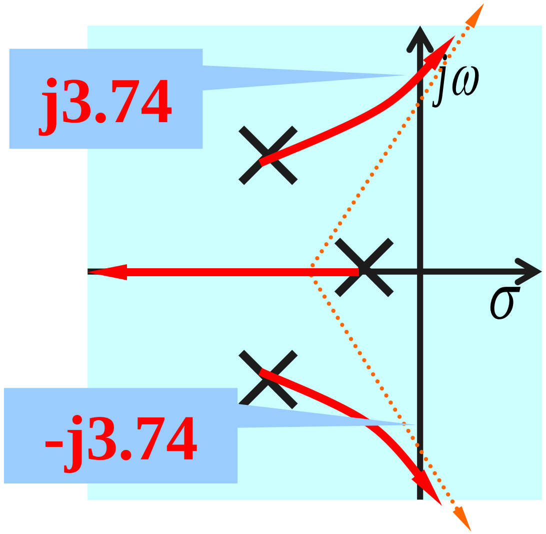
for real *s*.

Rule 7: The point where the locus crosses the imaginary axis

Rule #7: The point may be obtained by substituting *s=jω* into the characteristic equation and solving for *ω*.

Example :

$$G_k(s)=\frac{k}{(s+1)[(s+2)^2+6]}$$



Characteristic equation

$$(s+1)[(s+2)^2+6]+k=0$$

Substitute *s=jω*

$$(10+K-5\omega^2)-j(\omega^3-14\omega)=0$$

$$(10+K-5\omega^2)=0, \quad (\omega^3-14\omega)=0$$

$$\omega=\sqrt{14}=3.74, k=60$$

***s*=±*j*3.74**

(2) Utilize Routh's Stability Criterion

Characteristic equation: $s^3 + 5s^2 + 14s + (10 + k) = 0$

Routh array

s^3	**1**	**14**
s^2	**5**	**10+*k***
s^1	$\frac{70-(10+k)}{5}$ **=0**	
s^0	**10+*k***	

$5s^2 + 70 = 0$

$s_{1,2} = \pm j3.74$

***k* = 60**

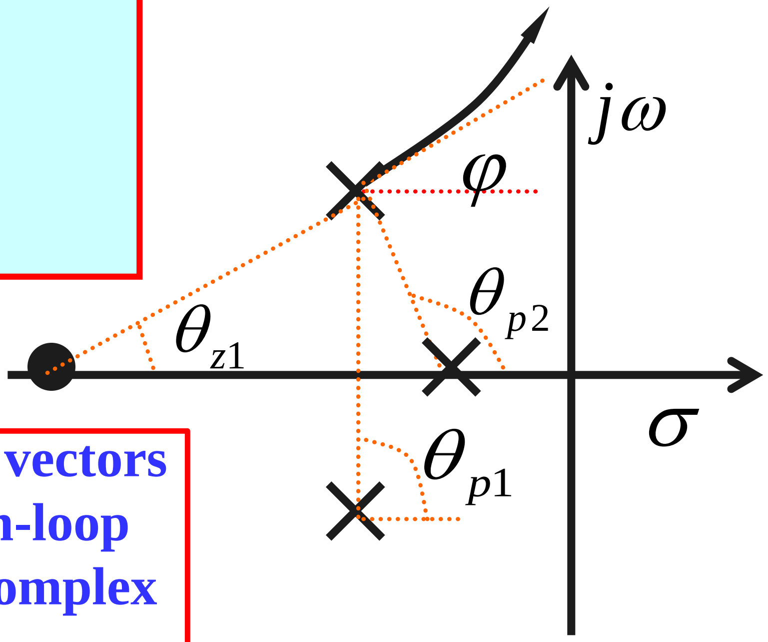
8. The angles of emergence and entry

The angle of emergence from complex poles is given by

$$\varphi_{pj} = \pm 180^{\circ}(2k+1) - \left(\sum \theta_{pj} - \sum \theta_{zi}\right)$$

Angles of the vectors from all other open-loop poles to the pole in question

Angles of the vectors from the open-loop zeros to the complex pole in question

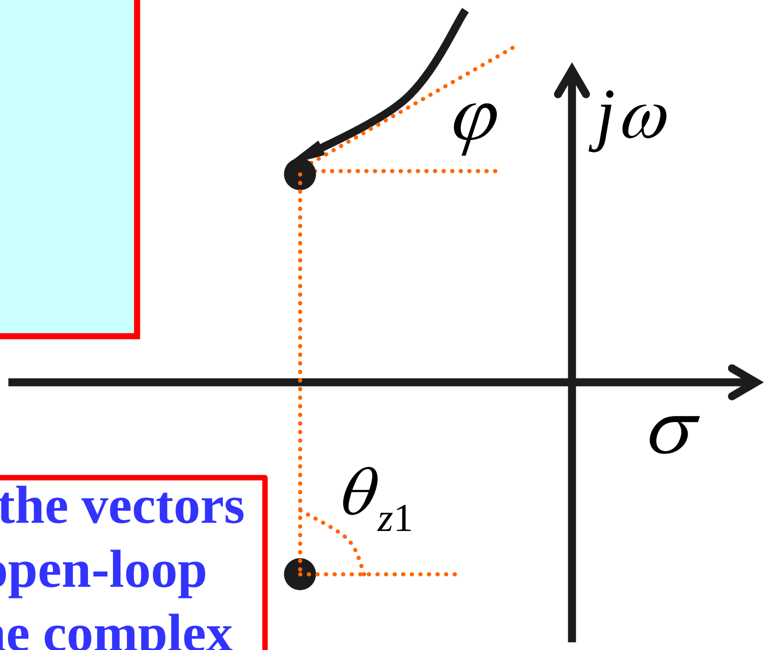


The angle of entry into a complex zero is given by

$$\varphi_{zi} = \pm 180^{\circ}(2k+1) - \left(\sum \theta_{z_i} - \sum \theta_{p_j}\right)$$

Angles of the vectors from all other open-loop zeros to the zero in question

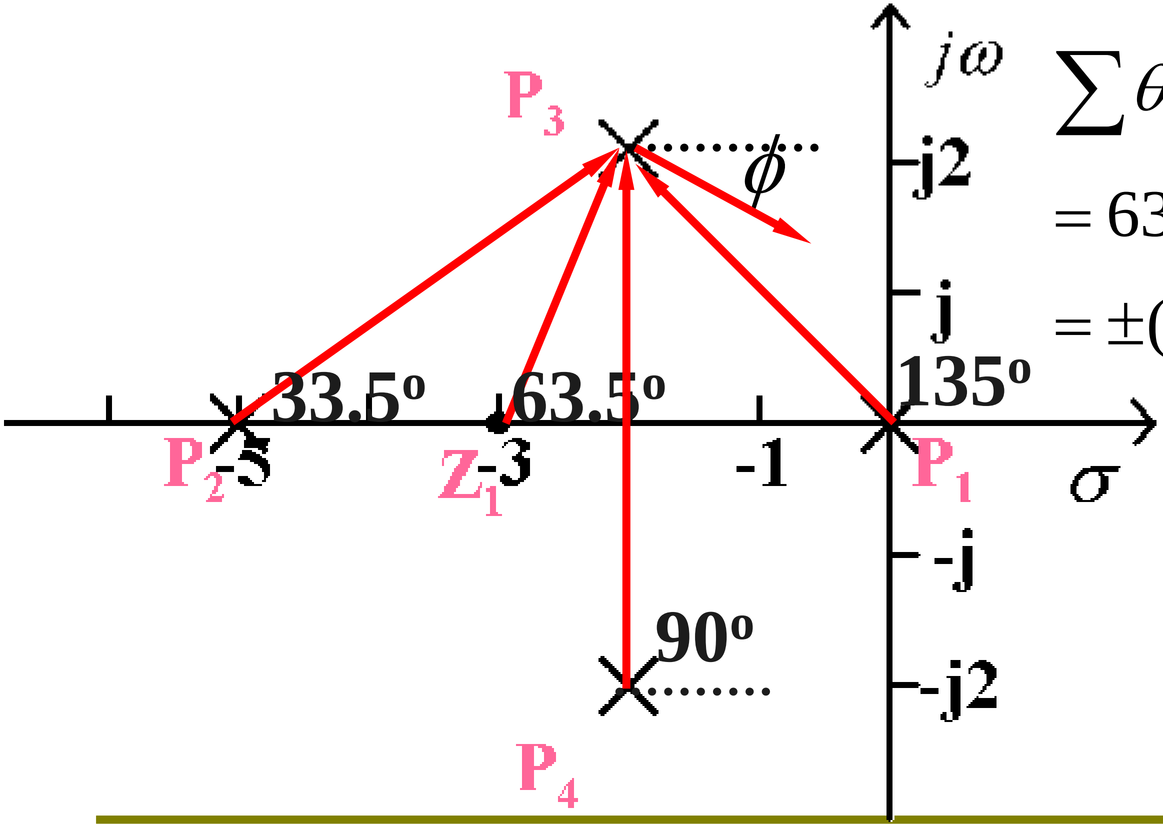
Angles of the vectors from the open-loop poles to the complex zero in question



Example: Given the open-loop transfer function

$$G(s)H(s)=\frac{K(s+3)}{s(s+5)[(s+2)^2+4]}$$

draw the angle of emergence from complex poles.



$$\sum\theta_{z_i}-\sum\theta_{p_j}$$

$$=63.5^{\circ}-(135^{\circ}+90^{\circ}+33.5^{\circ}+\phi)$$

$$=\pm(2k+1)180^{\circ}$$

$$\phi=-15^{\circ}\ \text{or}\ -375^{\circ}$$

Rule 9: The gain at a selected point s_t on the locus is obtained by applying Magnitude Equation

$$K_g = \frac{\prod_{j=1}^{n} |(s_t + p_j)|}{\prod_{i=1}^{m} |(s_t + z_i)|}$$

To locate a point with specified gain, use trial and error. Moving s_t toward the poles reduces the gain. Moving s_t away from the poles increases the gain.

Rule 10: The sum of real parts of the closed-loop poles is constant, independent of Kg, and equal to the sum of the real parts of the open-loop poles.

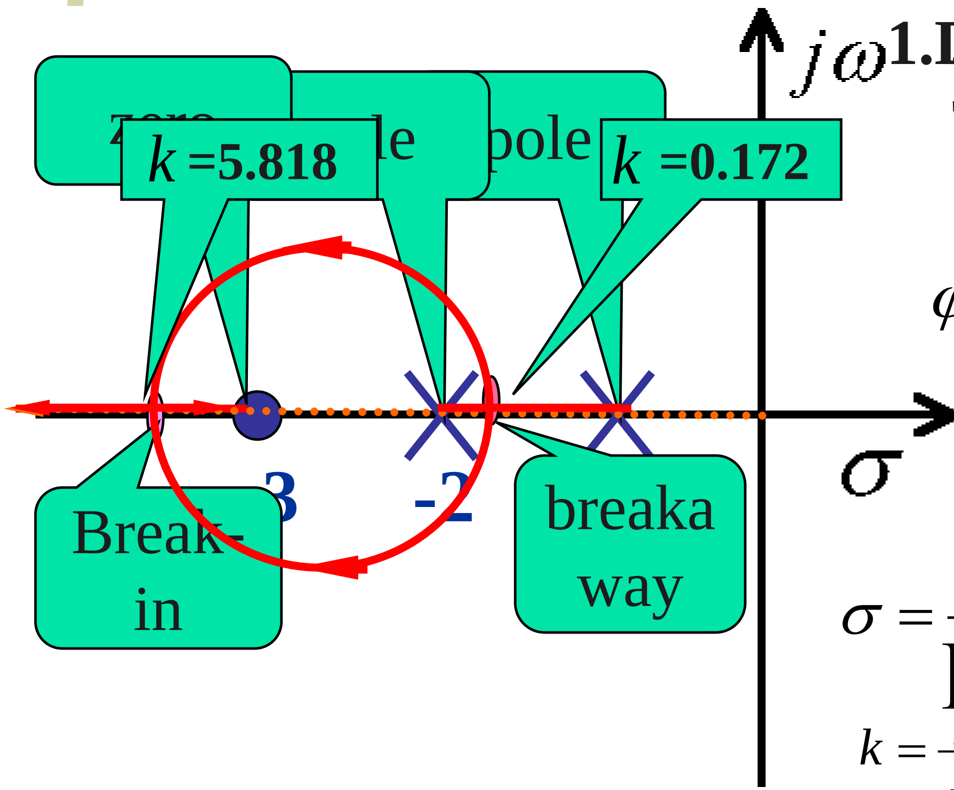
Summary

	Content	Rules
1	Continuity and Symmetry	Symmetry Rule
2	Starting and end points Number of segments	*n* segments start from n open-loop poles, and end at *m* open-loop zeros and *(n-m)* zeros at infinity.
3	Segments on real axis	On the left of an odd number of poles or zeros
4	Asymptote	***n-m*** **segments:** $\alpha = \frac{(2k+1)}{n-m}\pi, k = 0, \pm 1, \pm 2, K$
5	Asymptote	$\sigma = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{m}(-z_i)}{n-m}$

6	**Breakaway and break-in points**	$\frac{d[F(s)]}{ds}=0 \quad F(s)=P(s)+K_g Z(s)=0$ $P(s)\dot{Z}(s)-\dot{P}(s)Z(s)=0$ $\sum_{i=1}^{m}\frac{1}{z_i-\delta}=\sum_{j=1}^{n}\frac{1}{p_i-\delta}$
7	**Angle of emergence and entry**	**Angle of emergence** $\varphi_p=\mu\pi(2k+1)+\sum_{i=1}^{m}\theta_i-\sum_{\substack{j=1\\ j\neq p}}^{n}\varphi_j$ **Angle of entry** $\theta_z=\pm\pi(2k+1)+\sum_{j=1}^{n}\varphi_j-\sum_{\substack{i=1\\ i\neq z}}^{m}\theta_i$
8	**Cross on the imaginary axis**	**Substitute *s* = jω to characteristic equation and solve** **Routh's formula**

Example 5.2.1: Given the open-loop transfer function, please draw the root locus.

$$G(s)H(s)=\frac{k(s+3)}{(s+1)(s+2)}$$



1. Draw the open-loop poles and zeros

2. n, m

3. Symmetry

4. Segments on real axis

5. Asymptote

6. Breakaway and break-in points

$$\varphi=\frac{\pm180^{\circ}(2k+1)}{2-1}=\pm180^{\circ}$$

$$\sigma=\frac{\sum_{j=1}^{n}(-p_j)-\sum_{i=1}^{m}(-z_i)}{n-m}=\frac{(-1-2)+3}{2-1}=0$$

$$\frac{1}{3-a}=\frac{1}{1-a}+\frac{1}{2-a}$$

$$a=-3\pm\sqrt{2}$$

$$k=\frac{\prod_{j=1}^{n}|s+p_j|}{\prod_{i=1}^{m}|s+z_i|}=\frac{|a+1||a+2|}{|a+3|}$$

Example 5.2.2:

$$G(s)H(s)=\frac{k}{(s+1)^2(s+1+j18)(s+1-j18)}$$

1. Find poles and zeros

2. No segments on real axis

6. Breakaway and break-in

7. The points where the locus across the imaginary axis

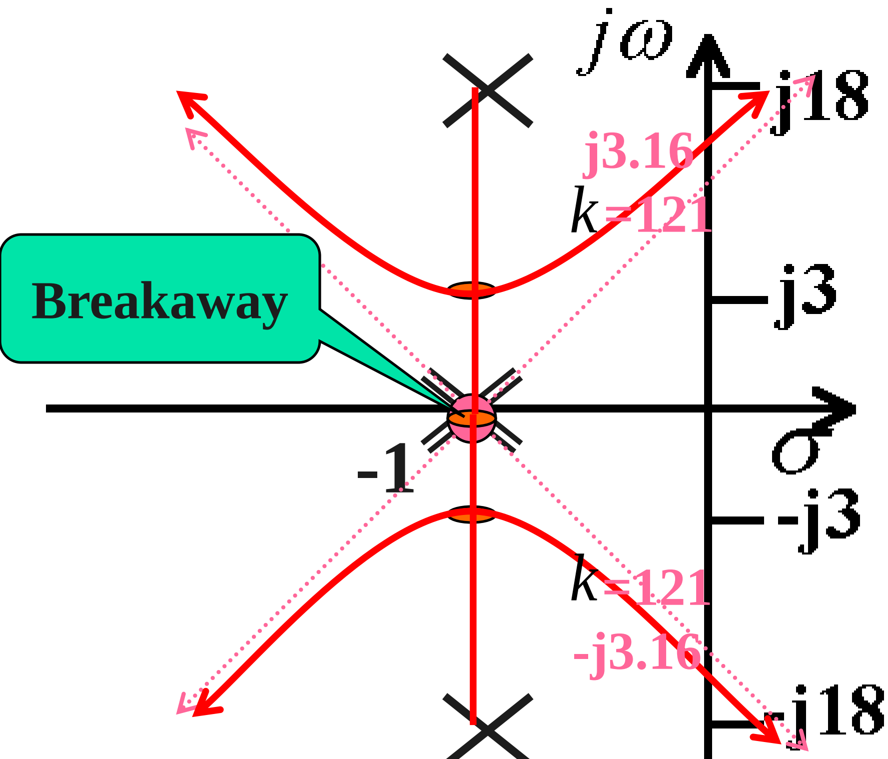
$$s^4+4s^3+24s^2+40s+19+k=0$$

s^4	1	24	$19+k$
s^3	4	40	0
s^2	14	$19+k$	
s^1	$\frac{484-4k}{14}$	0	
s^0	$19+k$		

$$14s^2+140=0$$

$$s=\pm j3.16$$

$484-4k=0$ get $k=121$



Example 5.2.3

$$G(s)H(s)=\frac{K}{s(s+1)(s+2)}$$

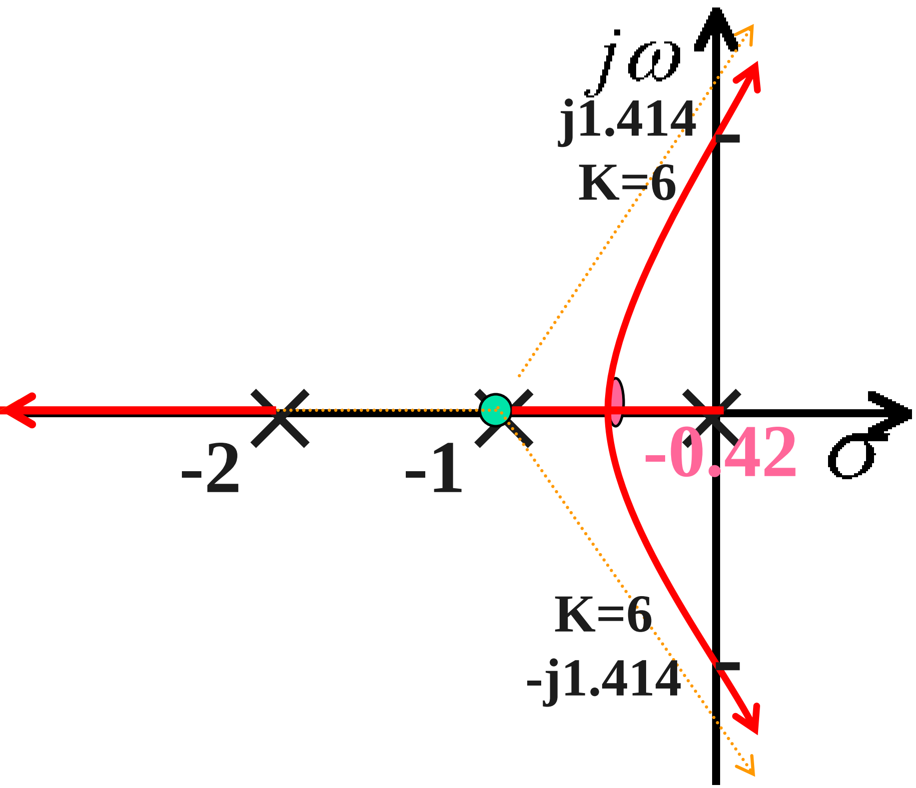
5. Points across the imaginary axis

1: Start from poles and zeros

2: Segments on the real axis

3: Asymptotes

4: Breakaway and break-in points



$$P(s)\dot{Z}(s)-\dot{P}(s)Z(s)=0$$

$$\frac{d}{ds}[s(s+1)(s+2)+K]=0$$

$$yields\ 3s^2+6s+2=0$$

$$solutions\ s_1=-0.42\quad s_2=-1.58$$

$$\sigma=-\frac{0+1+2}{3-0}=-1$$

$$\varphi=\pm\frac{180^\circ(2k+1)}{3-0}=\pm 60^\circ,180^\circ$$

$$s^3+3s^2+2s+K=0$$

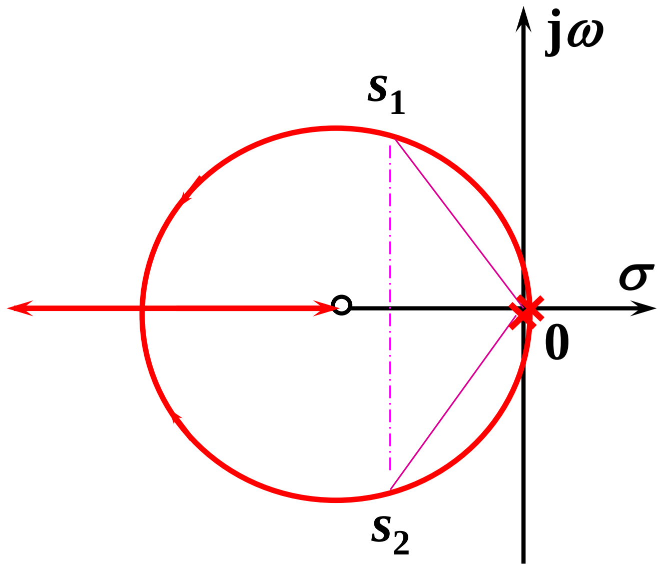
s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	=0
s^0	K	

$3s^2+6=0$

K=6

Example 5.2.4

$$G(s)=\frac{K^*(s+1)}{s^2}$$



Example 5.2.5