# Principle of Automatic Control

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**Slides and assignment can be downloaded from URL:**
https://piazza.com/sjtu.org/fall2017/ei304/resources

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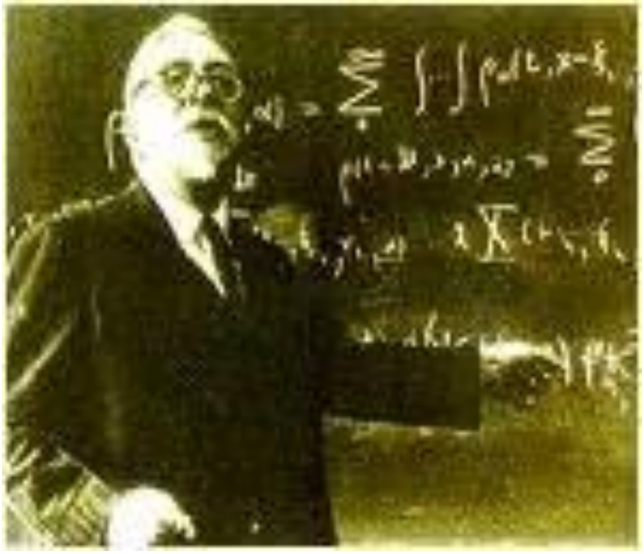
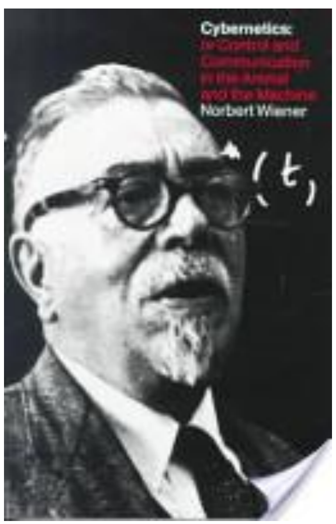


**Norbert Wiener (1894 –1964)**
**Harvard PhD, 1912 (18 years old)**

**America Mathematician**
**Professor of Mathematics at MIT**

**控制科学的鼻祖**



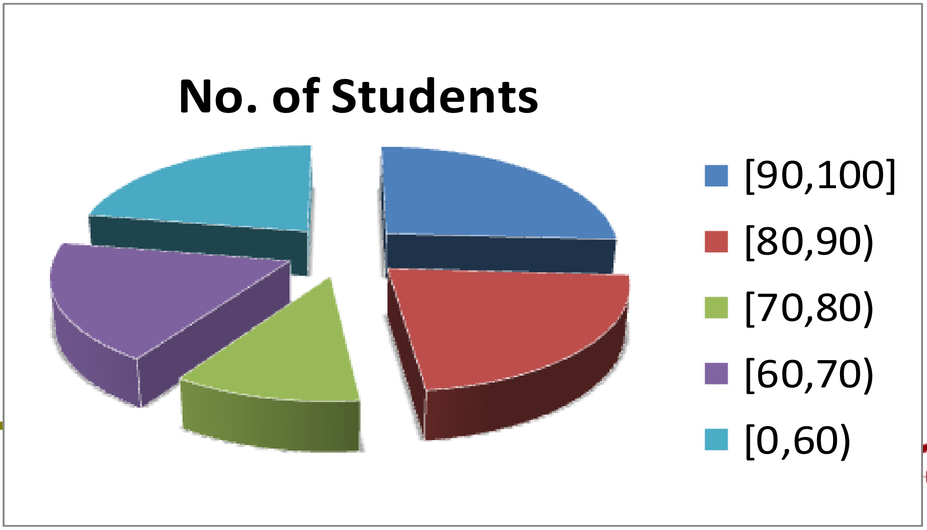

**Cybernetics-2$^{nd}$ Edition : or the Control and Communication in the Animal and the Machine**

**Wiener is regarded as the originator of cybernetics, a formalization of the notion of feedback, with many implications for engineering, systems control, computer science, biology, philosophy, and the organization of society.**

## Wiener: “Feedback is a method of controlling a system by inserting into it the result of its past performance”

- **Teaching and learning is a complex control system.**
- **Input:** **Teaching (course, tutorial, office meeting)**
- **Plant:** **Students**
- **Output:** **Outcome of learning**
- **Feedback—The key process for good output**

**Score of Quiz**



# Course Outline

| Week | Date | Content | Assignment |
|---|---|---|---|
| 9 | 07 Nov | Ch5: Root locus | |
| | 09 Nov | Ch5: Root locus | |
| 10 | 14 Nov | Ch5: Root locus | |
| | 16 Nov | Ch5: Root locus | |
| 11 | 21 Nov | Ch5: Root Locus<br>Ch6: Frequency response | Assign Due |
| | 23 Nov | Ch6: Frequency response | |
| 12 | 28 Nov | Ch6: Frequency response | |
| | 30 Nov | Ch6: Frequency response | |
| 13 | 05 Dec | Ch6: Frequency response | |
| | 07 Dec | Ch6: Frequency response<br>Ch7: Design of control systems | |

# Course Outline

| Week | Date | Content | Assignment |
|---|---|---|---|
| 14 | 12 Dec | Ch7: Design of control systems | Assign Due |
| | 14 Dec | Ch7: Design of control systems | |
| 15 | 17 Dec | Ch7: Design of control systems | |
| | 19 Dec | Ch7: Design of control systems | |
| 16 | 24 Dec | Review | Assign 7 Due |
| | 26 Dec | Review | |
| 17 | | Review (by yourself) | |
| 18 | | Final Exam (to be announced) | |

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# Chapter 5: Root Locus

# Contents

# 5-1 Introduction of Root Locus

## What we have known?

- We have had a quick look at **modeling and analysis** for specifying system performance.

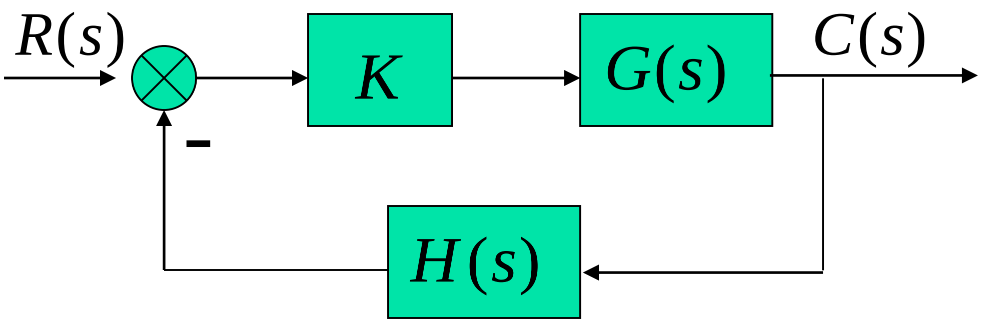
 (Time domain method for first and second order systems)

- We will now look at a **graphical approach**, known

as the root locus method, for analyzing and designing control systems

- As we will see, the root locations are important in determining the nature of the system response

# Proportional controller




**We have seen how this form of feedback is able to minimize the effect of disturbances.**

**For investigating the performance of a system, we have to solve the output response of the system.**

## Limitations:

**Root Locus**

**(1) It is difficult to solve the o … system – especially for a higher o…**

**(2) We can't easily investigate … of a system's performance from the time-domain**

**--especially when parameters of the system vary in a given range, or, when devices are added to the system.**

**History:**

**Root locus method was conceived by Evans in 1948.**

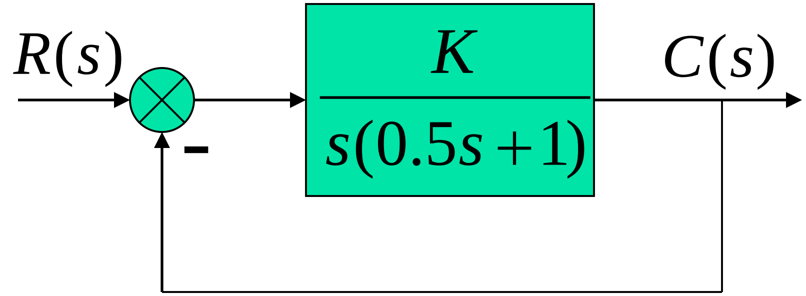
**Root locus method and Frequency Response method, which was conceived by Nyquist in 1938 and Bode in 1945, make up of the cores of the classical control theory for designing and analyzing control systems.**

**By using root locus method, we can:**

- **analyze the performance of the system**
- **determine the structure and parameter of the system**
- **design the compensator for control system**

**[Definition]:** **The path traced by the roots of the characteristic equation of the closed-loop system as the gain *K* varies from *0* to +∞ is called root locus.**

**Example:** The second order system with the open-loop transfer function




$$G_k(s) = \frac{K}{s(0.5s+1)}$$

Closed-loop Transfer function:

$$\Phi(s) = \frac{2K}{s^2+2s+2K}$$

Characteristic equation:

$$s^2+2s+2K=0$$

Roots of characteristic equation:

$$s_{1,2} = -1 \pm \sqrt{1-2K}$$

**Roots:** $s_{1,2} = -1 \pm \sqrt{1-2K}$

**[Discussion]:**

① **For *K=0*, $s_1=0$ and $s_2=-2$,**
**are the poles of open-loop system.**

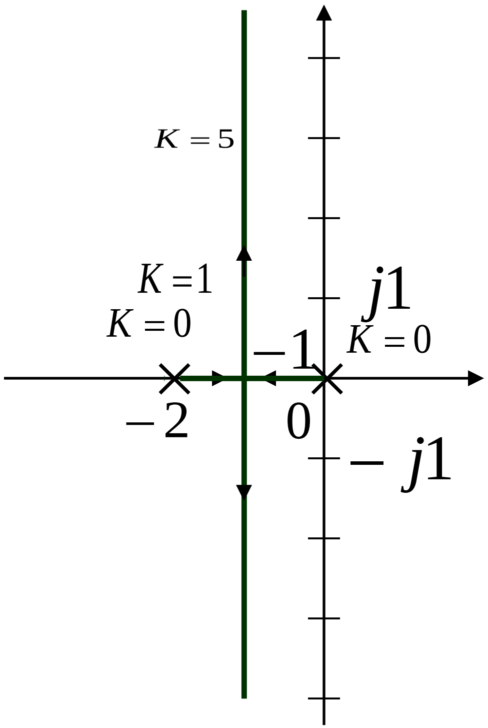
② **For *K=0.32*, $s_1=-0.4, s_2=-1.6$**

③ **For *K=0.5*, $s_1=-1, s_2=-1$**

④ **For *K=1*, $s_1=-1+j, s_2=-1-j$**

⑤ **For *K=5*, $s_1=-1+3j, s_2=-1-3j$**

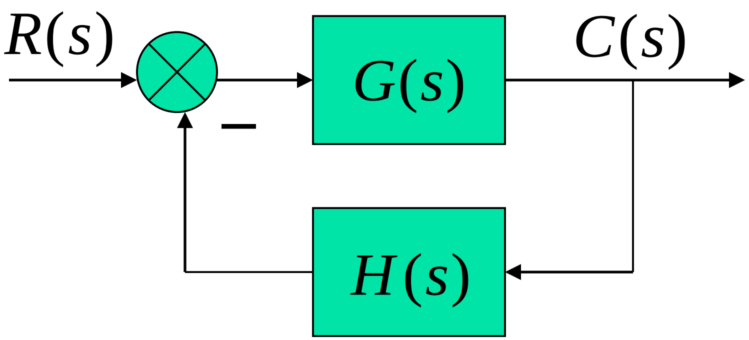
⑥ **For *K=∞*, $s_1=-1+\infty j, s_2=-1-\infty j$**




**[Terminology]: In the root locus, "×" denotes the poles of the open-loop transfer function, "●" zeros of the open-loop transfer function. The bode line represents the root locus, and arrow shows the root locus direction along some parameters.**

# 5-2 Properties of Root Locus

## Control System:




**Closed-loop transfer function:** $\Phi(s) = \dfrac{G(s)}{1+G(s)H(s)}$

**Open-loop transfer function:** $G_k(s) = G(s)H(s)$

**Canonical form of** $G_k(s)$

$$G_k(s) = K_g \cdot \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

$K_g$ – root locus gain

$-z_i$ and $-p_j$ are zeros and poles of open-loop transfer function

# 5.2.1 Two canonical form of open-loop transfer function

$$G_k(s) = G(s)H(s) = \frac{K\prod_{i=1}^{m}(\tau_i s + 1)}{\prod_{j=1}^{n}(T_j s + 1)} \qquad n \geq m \qquad (5\text{-}1)$$

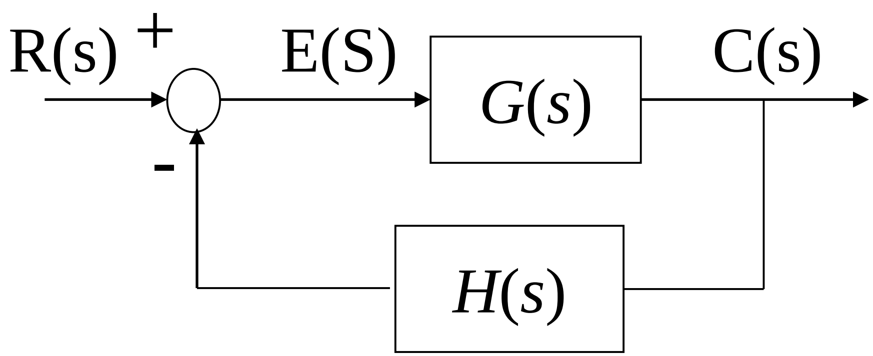
**Time constant canonical form: *K* is open-loop gain**

$$= \frac{K_g\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} \qquad K = K_g\frac{\prod_{i=1}^{m} z_i}{\prod_{j=1}^{n} p_j} \qquad (5\text{-}2)$$

**Zero-pole canonical form: $K_g$ is root locus gain**

**In root locus method, we use (5-2）.**

# 5.2.2 Magnitude and Phase Equations




**Characteristic equation of closed-loop transfer function**

$$1 + G(s)H(s) = 0 \qquad G(s)H(s) = -1 \qquad (4\text{-}3)$$

**The objective is to determine values of *s* that satisfy the equation. Such *s* are the on the root locus.**

# Definition

$$G_k(s) = -1 \text{ or } k_g \cdot \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = -1$$ **is the root-locus equation.**

**Remembering that *s* may be complex, the root-locus equation can be equivalently written into**

$$K_g \cdot \frac{\prod_{i=1}^{m}|(s+z_i)|}{\prod_{j=1}^{n}|(s+p_j)|} = 1$$

**Magnitude equation**

$$\angle\sum_{i=1}^{m}(s+z_i) - \angle\sum_{j=1}^{n}(s+p_j) = \pm(2k+1)\pi, k = 0,1,2\ldots$$

**Phase equation**
**Argument equation**

**Root locus method is performed in two stages:**

1. **Finding all values of *s* satisfying the argument equation**
2. **Finding particular values of *s* that satisfy the magnitude equation**

**Phase equation is the sufficient and necessary condition for root locus.**

**—— All the values of *s* satisfying the argument equation constitute the root locus.**

**——Magnitude equation is used to determine the gain $K_g$ for a particular root.**

**——Argument equation is independent on $K_g$.**

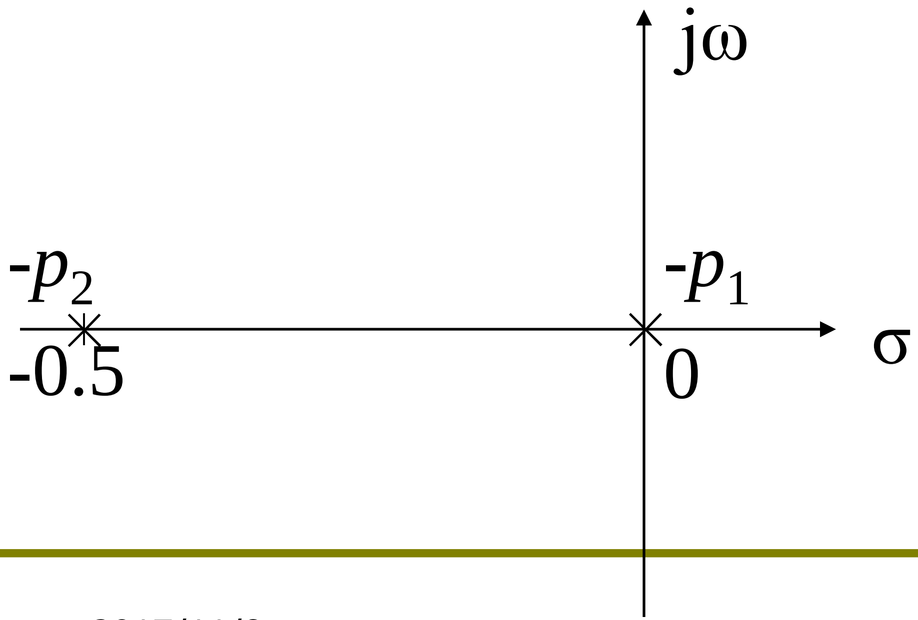
# Trial guess method:

Example 5.1 $G_k(s)=\frac{k}{s(2s+1)}$

Solution:

**1) Find zeros and poles of open-loop transfer function on *s* plane**

$$G_k(s)=\frac{0.5k}{s(s+0.5)}=\frac{K_g}{s(s+0.5)},\quad K_g=0.5k$$




**Poles:**

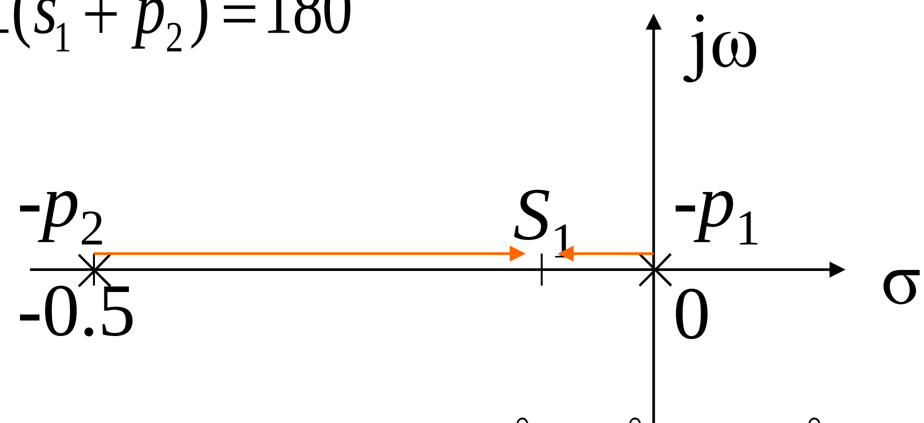
$$-p_1=0,\quad -p_2=-0.5$$

**There are no zeros.**

## 2）Inspecting the real axis：

$s_1>0$ $\quad \angle(s_1+p_1)=\angle(s_1+p_2)=0^\circ$ **Argument equation is not satisfied.**

$-\infty< s_1 <-p_2$ $\quad \angle(s_1+p_1)=\angle(s_1+p_2)=180^\circ$




$-p_2< s_1 <-p_1$ $\quad \angle(s_1+p_1)+\angle(s_1+p_2)=180^\circ+0^\circ=180^\circ$

**The all values between $-p_1 \sim -p_2$ of the real axis are the points of root locus**

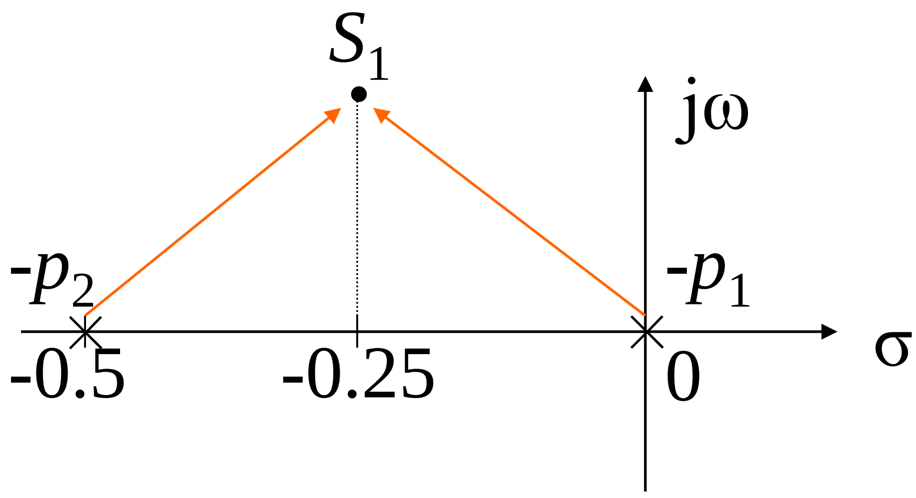
**For $s_1$=-0.1** $\quad K_g=|s_1+p_1|\cdot|s_1+p_2|=0.1\times0.4=0.04$

## 3）Examine locations off the real axis: arbitrarily choose an $s_1$

**In order to satisfy**

$$-\angle(s_1+p_1)-\angle(s_1+p_2)=-180^\circ$$

**$s_1$ should be on the perpendicular bisector of the line between the two poles.**



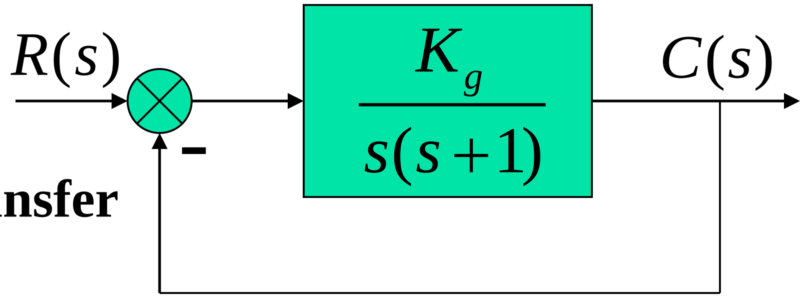

Let $s_1=-0.25+j0.25$

$$-\angle(s_1+p_1)-\angle(s_1+p_2)=-135^0-45^0=-180^0$$

$$K_g=|s_1+p_1|\cdot|s_1+p_2|=\left|0.25\times\sqrt{2}\right|^2=0.125$$

**Example: For the second order system，draw the root locus with respect to $K_g$ from $0 \rightarrow \infty$**

**Solution:** $$\Phi(s) = \frac{K_g}{s^2 + s + K_g}$$



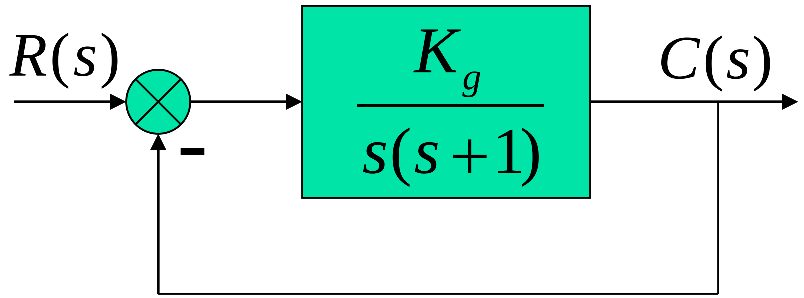

**Characteristic equation of closed-loop transfer function:**

$$s^2 + s + K_g = 0, s_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4K_g}$$

**[Discussion]:**

① For $K_g = 0$, we have $s_{1,2} = 0$ and -1

② $K_g \uparrow$, $s_1$ **goes to the left along the negative real axis, and** $s_2$ **goes to the right along negative real axis from -1.**




③ For $K_g = \frac{1}{4}, s_{1,2} = -\frac{1}{2}$

For any $0 < K_g < \frac{1}{4}$, $s_{1,2}$ are on the negative real axis.

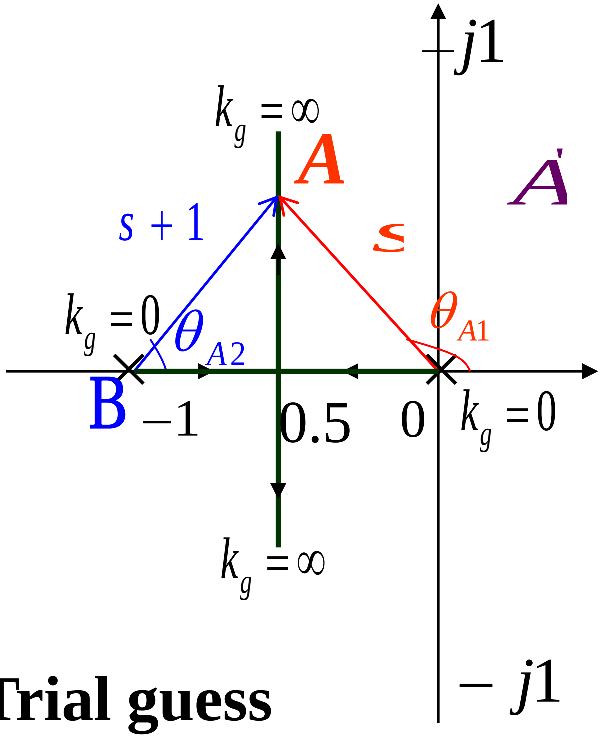
④ For $K_g > \frac{1}{4}$, $s_{1,2}$ are complex numbers

**As $K_g$ increases, location of roots branch out along the vertical line.**

⑤ $K_g \to \infty,\ s_{1,2} = -\frac{1}{2} \pm j\infty$

**[Summary]**

When $K_g$ changes from zero to infinity, there are two segments starting from the open-loop poles to infinity.




For higher order system, it is very difficult to sketch all the root loci. But graphical method shows its advantage.

Are Points A and A' on the root locus? (Trial guess method)

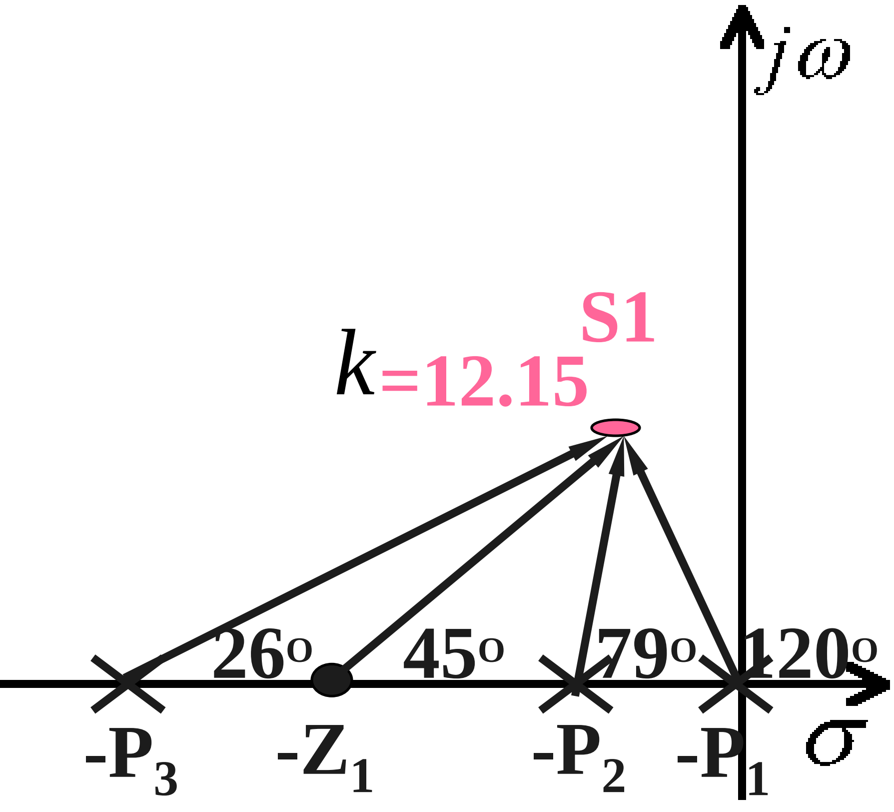
$$\angle \frac{k_g}{s(s+1)} = -\angle s - \angle (s+1) = -\angle \vec{OA} - \angle \vec{BA} = \theta_{A1} + \theta_{A2} = \pm(2k+1)\pi$$

Obviously, A is on the root locus, but A' is not on the root locus.

## Example 5-2: The open-loop transfer function is given by

$$G(s)H(s)=\frac{k(s+4)}{s(s+2)(s+6.6)}$$

**Check if $s_1$= -1.5+*j*2.5 is on the root locus. If yes, please determine the parameter gain *k***




**1)Via argument equation**

$$\angle(s_1+z_1)-\angle(s_1+p_1)-\angle(s_1+p_2)-\angle(s_1+p_3)$$

$$=45°-120°-79°-26°=-180°$$

**2)Via magnitude equation**

$$k=\frac{|s_1+p_1|\cdot|s_1+p_2|\cdot|s_1+p_3|}{|s_1+z_1|}$$

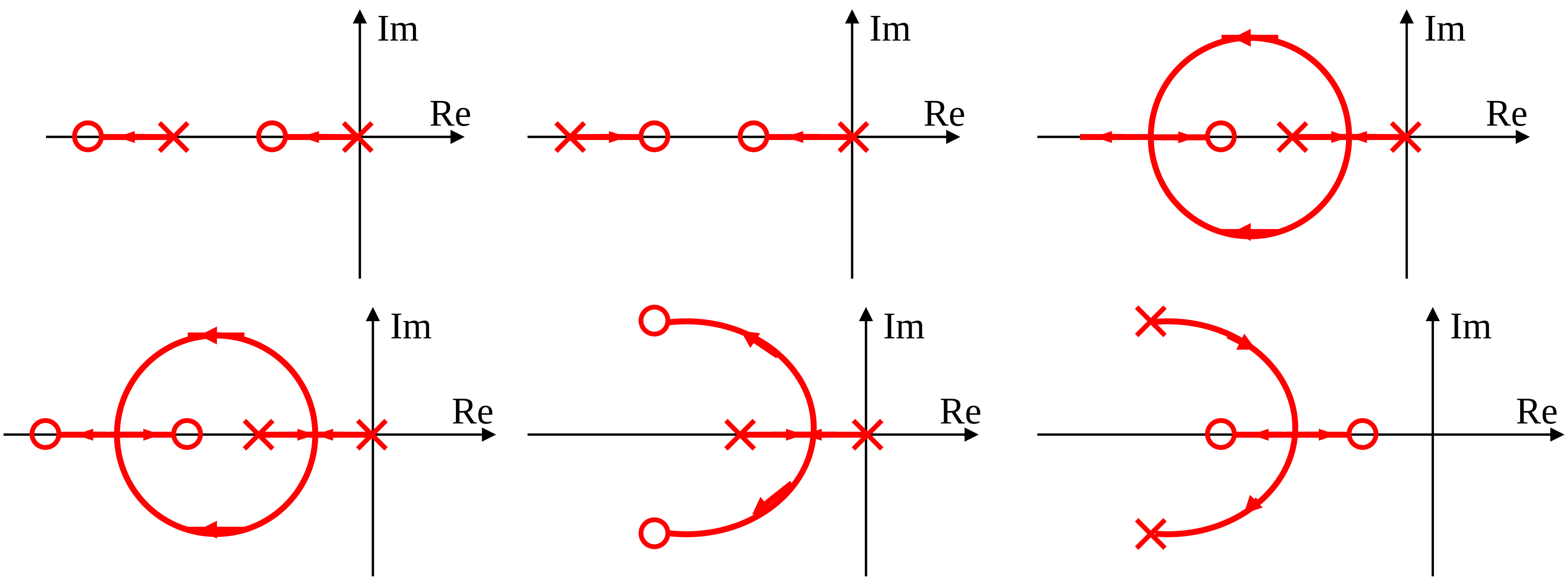
$$=\frac{2.9\times2.6\times5.8}{3.6}=12.15$$

## We can rapidly sketch the root-loci of a control system.

**Sketch the root-loci for the following open-loop transfer functions:**

(1) $G(s)=\dfrac{K(s+2)(s+6)}{s(s+4)}$  (2) $G(s)=\dfrac{K(s+2)(s+4)}{s(s+6)}$  (3) $G(s)=\dfrac{K(s+3)}{s(s+2)}$

(4) $G(s)=\dfrac{K(s+2)(s+3)}{s(s+1)}$  (5) $G(s)=\dfrac{K(s^2+6s+13)}{s(s+2)}$  (6) $G(s)=\dfrac{K(s+1)(s+2)}{(s^2+6s+13)}$



# Summary

- The preceding angle and magnitude criteria can be used to verify which points in the *s-plane form part of the root locus*
- It is not practical to evaluate all points in the s-plane to find the root locus
- We can formulate a number of rules that allow us to sketch the root locus